

Approximately Diagonalizing Matrices Over $C(Y)$

Huaxin Lin

Abstract

Let X be a compact metric space which is locally absolutely retract and let $\varphi : C(X) \rightarrow C(Y, M_n)$ be a unital homomorphism, where Y is a compact metric space with $\dim Y \leq 2$. It is proved that there exists a sequence of n continuous maps $\alpha_{i,m} : Y \rightarrow X$ ($i = 1, 2, \dots, n$) and a sequence of sets of mutually orthogonal rank one projections $\{p_{1,m}, p_{2,m}, \dots, p_{n,m}\} \subset C(Y, M_n)$ such that

$$\lim_{m \rightarrow \infty} \sum_{i=1}^n f(\alpha_{i,m}) p_{i,m} = \varphi(f) \text{ for all } f \in C(X).$$

This is closely related to the Kadison diagonal matrix question. It is also shown that this approximate diagonalization could not hold in general when $\dim Y \geq 3$.

1 Introduction

Over two decades ago, Richard Kadison proved that a normal element in $M_n(\mathcal{N})$, where \mathcal{N} is a von-Neumann algebra, can be diagonalized ([7] and [8]). He showed that this cannot be done if \mathcal{N} is replaced by a unital C^* -algebra in general. He then asked what topological properties of a compact metric space Y will guarantee that every normal element in $M_n(C(Y))$ can always be diagonalized. Karsten Grove and Gert K. Pedersen [9] showed that this could not go very far. They demonstrated that Y has to be sub-Stonean and $\dim Y \leq 2$ if every self-adjoint element can be diagonalized in $M_n(C(Y))$. Furthermore, they showed that, even for sub-Stonean spaces Y with $\dim Y \leq 2$, one still could not diagonalize a normal element in general. In fact, they showed that in order to have every normal element in $M_n(C(Y))$ to be diagonalized, one must have that every finite covering space over each closed subset of Y is trivial and every complex line bundle over each closed subset of Y is trivial, in addition to the requirements that X is sub-Stonean and $\dim Y \leq 2$. So not every sub-Stonean space X with dimension at most two has the property that every normal element can be diagonalized. Since sub-Stonean spaces are not the every-day topological space with dimension at most two, it seems that the question of diagonalizing normal elements in $M_n(C(Y))$ has a rather negative answer.

However, in the decades after the original question was raised and answered, it seems that approximately diagonalizing some normal elements or some commutative C^* -subalgebras in $M_n(C(Y))$, where Y is a lower dimensional nice topological space, becomes quite useful and important. In this paper, instead of considering exact diagonalization of commutative C^* -subalgebras in $M_n(C(Y))$, we study the problem whether a unital homomorphism $\varphi : C(X) \rightarrow M_n(C(Y))$ can be approximately diagonalized. To be precise, we formulate as follows: Let $\epsilon > 0$ and a compact set $\mathcal{F} \subset C(X)$ be given. Are there continuous maps $\alpha_i : Y \rightarrow X$ ($1 \leq i \leq n$) and mutually orthogonal rank one projections $p_1, p_2, \dots, p_n \in M_n(C(Y))$ such that

$$\|\varphi(f) - \sum_{i=1}^n f(\alpha_i) p_i\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e1.1})$$

Here rank one projections are those projections in $M_n(C(Y))$ for which every fiber has rank one. Note that we do not insist that p_1, p_2, \dots, p_n are equivalent. Note also that if $a \in M_n(C(Y))$ is a normal element and if $sp(a) = X$, a compact subset of the plane, then a induces a unital homomorphism $\varphi : C(X) \rightarrow M_n(C(Y))$ by defining $\varphi(f) = f(a)$ for all $f \in C(X)$. However, we study the general case that X is a compact metric space.

It has been known that those unital homomorphisms which can be approximately diagonalized are very useful, for example, in the study of inductive limits of homogeneous C^* -algebras, a subject has profound impact in the program of classification of amenable C^* -algebras, otherwise known as the Elliott program.

The main result that we report here is that the answer to (e 1.1) is affirmative for any compact metric space X which is locally absolutely retract (see 6.1) and any compact metric space Y with $\dim Y \leq 2$. Moreover, we show that, the answer is negative for general compact metric space Y with $\dim Y \geq 3$. In fact, a unitary in $M_2(C(S^3))$ may not be approximately diagonalized. We also show that if $\dim Y > 3$, then there are self-adjoint elements with spectrum $[0, 1]$ in $M_n(C(Y))$ which can not be approximately diagonalized. For more general compact metric space X , we show that, for any $\epsilon > 0$, any compact subset $\mathcal{F} \subset C(X)$, there is a unital commutative diagonal C^* -subalgebra $B \subset M_n(C(Y))$ such that

$$\text{dist}(\varphi(f), B) < \epsilon \text{ for all } f \in \mathcal{F},$$

provided that $\dim Y \leq 2$ (see 6.5).

As expected, when one studies unital homomorphisms from $C(X) \rightarrow M_n(C(Y))$, one often needs to make some perturbation. The trouble arises when one tries to approximate an ‘almost homomorphism’ by a homomorphism. This is already problematic when Y is just a point. Suppose that X is a compact subset of the plane and $L : C(X) \rightarrow M_n$ is unital positive linear map which is almost multiplicative. It was first proved by W.A.J. Luxembour and F.R. Taylor [14], using non-standard analysis, that such maps can be approximated by homomorphisms. Theorem 2.2 generalizes this to the case that X is any compact metric space. Theorem 5.1 shows that the same statement holds when M_n is replaced by $C([0, 1], M_n)$.

The paper is organized as follows. In section two, we present a theorem which generalizes the early result of Luxembour and Taylor mentioned above. In section three, we collect some easy facts. Lemma 3.1 serves as a uniqueness theorem in finite dimensional C^* -algebras and Lemma 3.3 may be viewed as an elementary version of the so-called Basic Homotopy Lemma (see [1] and [12]). In section four, we provide a uniqueness theorem for homomorphisms from $C(X)$ into $C([0, 1], M_n)$ which extends Lemma 3.1. In section five, using the results in the previous sections, we first show that an ‘almost homomorphism’ from $C(X)$ into $C([0, 1], M_n)$ is close to a true homomorphism, further generalizing the theorem of Luxembour and Taylor. Then we present a version of Basic Homotopy Lemma in $C([0, 1], M_n)$. Section six contains the main result which based on the previous sections, in particular, the uniqueness theorem in section four and the Basic Homotopy Lemma in section five. Finally, in section seven, we show that one should not expect same results when $\dim Y \geq 3$.

Acknowledgements: Most of this work was done when the author was in East China Normal University in the summer 2009. This work was partially supported by a NSF grant, Changjiang Professorship from ECNU and Shanghai Priority Academic Disciplines.

2 Approximate homomorphisms

Lemma 2.1. *Let X be a compact metric space and let $n \geq 1$ be an integer. Let $H : C(X) \rightarrow C^b(\mathbb{N}, M_n)/C_0(\mathbb{N}, M_n)$ be a unital homomorphism. Then there exists an infinite subsequence*

$S \subset \mathbb{N}$ such that the induced homomorphism $H' : C(X) \rightarrow C^b(S, M_n)/C_0(S, M_n)$ has finite spectrum.

Proof. Put $A_0 = C^b(\mathbb{N}, M_n)/C_0(\mathbb{N}, M_n)$. Denote by $\pi : C^b(\mathbb{N}, M_n) \rightarrow A_0$ the quotient map. Let $\xi_1 \in X$ be a point in the spectrum of H . Let

$$I_1 = \{f \in C(X) : f(\xi_1) = 0\}.$$

Then $\overline{H(I_1)A_0H(I_1)}$ is a σ -unital hereditary C^* -subalgebra and it is not A_0 , since ξ_1 is in the spectrum of H and $H(I_1)$ is a proper closed ideal of $H(C(X))$. Note that A_0 is the corona algebra of the separable C^* -algebra $C_0(\mathbb{N}, M_n)$. It follows from a theorem of G. Pedersen (Th.15 of [16]) that $\overline{H(I_1)A_0H(I_1)}^\perp \neq \{0\}$. Since $\overline{H(I_1)A_0H(I_1)}^\perp$ is a hereditary C^* -subalgebra of A_0 and A_0 has real rank zero, there is a non-zero projection $p_1 \in \overline{H(I_1)A_0H(I_1)}^\perp$. It follows that

$$H(f)p_1 = f(\xi_1)p_1 \text{ for all } f \in C(X). \quad (\text{e 2.2})$$

It is standard that there exists a sequence of projections $\{p_1(m)\} \subset M_n$ such that $\pi(\{p_1(m)\}) = p_1$. Let $S_1 \subset \mathbb{N}$ be the subsequence so that $p_1(m) \neq 0$ for all $m \in S_1$. Note that S_1 must be infinite. Let $A_1 = A_0/J_1$, where

$$J_1 = \{\{a_m\} \in C^b(\mathbb{N}, M_n) : a_m = 0 \text{ for all } m \in S_1\}.$$

One also has that $A_1 \cong C^b(S_1, M_n)/C_0(S_1, M_n)$. Let $\Phi_1 : A_0 \rightarrow A_1$ be the quotient map and define $H_1 = \Phi_1 \circ H$. If ξ_1 is the only point in the spectrum of H_1 , the lemma follows. Otherwise, let $\xi_2 \neq \xi_1$ be another point in the spectrum of H_1 . Let

$$I_2 = \{f \in C(X) : f(\xi_2) = 0\}.$$

From the above argument, one obtains a nonzero projection $p_2 \in \overline{H_1(I_2)A_1H_1(I_2)}^\perp$. Then $\Phi_1(p_1)p_2 = 0$ and

$$H_1(f)p_2 = f(\xi_2)p_2 \text{ for all } f \in C(X). \quad (\text{e 2.3})$$

There exists a projection $\{p_2(m)\} \in C^b(S_1, M_n)$ such that $\pi_1(\{p_2(m)\}) = p_2$ and

$$p_2(m)p_1(m) = 0 \quad (\text{e 2.4})$$

for all m , where $\pi_1 : C^b(S_1, M_n) \rightarrow A_1$ is the quotient map. Let $S_2 \subset S_1$ be such that $p_2(m) \neq 0$ for all $m \in S_2$. Then S_2 is an infinite subset. Let

$$J_2 = \{\{a_m\} \in C^b(S_1, M_n) : a_m = 0 \text{ for all } m \in S_2\}.$$

Put $A_2 = A_1/J_2$ and let $\Phi_2 : A_1 \rightarrow A_2$ be the quotient map. Note that $A_2 \cong C^b(S_2, M_n)/C_0(S_2, M_n)$. Moreover,

$$p_1(m) \neq 0 \text{ and } p_2(m) \neq 0 \text{ for all } m \in S_2. \quad (\text{e 2.5})$$

Define $H_2 = \Phi_2 \circ H_1$. Then ξ_1, ξ_2 are in the spectrum of H_2 . If the spectrum of H_2 contains only these two points, the lemma follows. Otherwise, we continue. However, since there can be no more than n mutually orthogonal non-zero projections in M_n , from (e 2.4) and (e 2.5), this process has to stop at the stage n or earlier. At that point, one obtains an infinite subset $S \subset \mathbb{N}$, for which $H' : C(X) \rightarrow C^b(S, M_n)/C_0(S, M_n)$ has finite spectrum. \square

Theorem 2.2. *Let X be a compact metric space, let $n \geq 1$ be an integer and let $M > 0$. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: for any unital map $\varphi : C(X) \rightarrow M_n$ with $\|\varphi(f)\| \leq M$ for all $f \in C(X)$ with $\|f\| \leq 1$ such that*

$$\|\varphi(\lambda_1 x + \lambda_2 y) - (\lambda_1 \varphi(x) + \lambda_2 \varphi(y))\| < \delta, \quad (\text{e 2.6})$$

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| < \delta \text{ and } \|\varphi(x^*) - \varphi(x)^*\| < \delta \quad (\text{e 2.7})$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_i| \leq 1$ ($i = 1, 2$) and $x, y \in \mathcal{G}$, there exists a unital homomorphism $\psi : C(X) \rightarrow M_n$ such that

$$\|\varphi(f) - \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{G}. \quad (\text{e 2.8})$$

Proof. Let $H(C(X), M_n)$ be the set of unital homomorphisms. Suppose that the theorem fails. There exists $\epsilon_0 > 0$ and a finite subset $\mathcal{G}_0 \subset C(X)$ with the following properties:

There exists a sequence of unital maps $\varphi_k : C(X) \rightarrow M_n$ with $\|\varphi_k(f)\| \leq M$ for all $\|f\| \leq 1$ ($k = 1, 2, \dots$) such that

$$\lim_{k \rightarrow \infty} \|\varphi_k(\lambda_1 x + \lambda_2 y) - (\lambda_1 \varphi_k(x) + \lambda_2 \varphi_k(y))\| = 0, \quad (\text{e 2.9})$$

$$\lim_{k \rightarrow \infty} \|\varphi_k(xy) - \varphi_k(x)\varphi_k(y)\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|\varphi_k(x^*) - \varphi_k(x)^*\| = 0 \quad (\text{e 2.10})$$

for all $x, y \in C(X)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_i| \leq 1$, $i = 1, 2$, but

$$\inf_{\psi \in H(C(X), M_n)} \left\{ \inf_k \sup \{ \|\varphi_k(f) - \psi(f)\| : f \in \mathcal{G}_0 \} \right\} \geq \epsilon_0, \quad (\text{e 2.11})$$

where $H(C(X), M_n)$ is the set of unital homomorphisms from $C(X)$ to M_n . Let $A = C^b(\mathbb{N}, M_n)$, $I = C_0(\mathbb{N}, M_n)$ and let $\pi : A \rightarrow A/I$ be the quotient map. Define $L : C(X) \rightarrow A$ by $L(f) = \{\varphi_k(f)\}_{k \in \mathbb{N}}$ for $f \in C(X)$ and $H = \pi \circ L$. From (e 2.9) and (e 2.10), $H : C(X) \rightarrow A/I$ is a unital homomorphism. It follows from 2.1 that there exists an infinite subset $S \subset \mathbb{N}$ such that $H_1 : C(X) \rightarrow C^b(S, M_n)/C_0(S, M_n)$ defined by $H_1 = \Phi \circ H$ has finite spectrum, where $\Phi : A \rightarrow A_1 = A/I_1$ is the quotient map and where

$$I_1 = \{ \{a_m\} \in C^b(\mathbb{N}, M_n) : a_m = 0 \text{ for all } m \in S \}.$$

By passing to a subsequence, without loss of generality, one may assume that H has finite spectrum. Therefore there are mutually orthogonal projections $\{p_1, p_2, \dots, p_K\} \subset A$ such that

$$H(f) = \sum_{j=1}^K f(\xi_j) p_j \text{ for all } f \in C(X), \quad (\text{e 2.12})$$

where $\{\xi_1, \xi_2, \dots, \xi_K\} \subset X$ is a finite subset. There are mutually orthogonal projections $P_1, P_2, \dots, P_K \in C^b(\mathbb{N}, M_n)$ such that $\pi(P_j) = p_j$, $j = 1, 2, \dots, K$. Let $P_j = \{q_j(m)\}$, where each $q_j(m)$ is a projection and $q_j(m)q_i(m) = 0$, if $i \neq j$, $i, j = 1, 2, \dots, K$. Define $\psi_m : C(X) \rightarrow M_n$ by $\psi_m(f) = \sum_{j=1}^K f(\xi_j) q_j(m)$ for all $f \in C(X)$. Then $\pi \circ \{\psi_m\} = H$. It follows that

$$\lim_{m \rightarrow \infty} \|\varphi_m(f) - \psi_m(f)\| = 0 \text{ for all } f \in C(X). \quad (\text{e 2.13})$$

Hence, there exists an integer $N \geq 1$ such that $\|\varphi_m(f) - \psi_m(f)\| < \epsilon_0/2$ for all $f \in \mathcal{F}$. This contradicts with (e 2.11). The lemma follows. \square

Corollary 2.3. *Let $k, n \geq 1$ be two integers and let $\epsilon > 0$. Then there exists $\delta > 0$ satisfying the following: Suppose that $x_1, x_2, \dots, x_k \in M_n$ are k self-adjoint elements with $\|x_i\| \leq 1$ ($i = 1, 2, \dots, k$) for which*

$$\|x_i x_j - x_j x_i\| < \delta, \quad i = 1, 2, \dots, k.$$

Then there are k self-adjoint elements $y_1, y_2, \dots, y_k \in M_n$ with $\|y_i\| \leq 1$ ($i = 1, 2, \dots, k$) such that

$$y_i y_j = y_j y_i \quad \text{and} \quad \|x_i - y_i\| < \epsilon, \quad i = 1, 2, \dots, k. \quad (\text{e 2.14})$$

Proof. This follows from 2.1 as in the proof of 2.2. One sketches here. If the corollary fails, there would be an $\epsilon_0 > 0$ and a sequence of k self-adjoint elements $\{x_j^{(m)}\}$, $j = 1, 2, \dots, k$, such that

$$\lim_{m \rightarrow \infty} \|x_j^{(m)} x_i^{(m)} - x_i^{(m)} x_j^{(m)}\| = 0, \quad i, j = 1, 2, \dots, k, \quad \text{and} \quad (\text{e 2.15})$$

$$\inf_m \{ \inf_m \{ \max \{ \|x_j^{(m)} - y_j\| : 1 \leq j \leq m \} \} \} \geq \epsilon_0, \quad (\text{e 2.16})$$

where the outside infimum is taken among all possible commuting k -tuple of self-adjoint elements $\{y_j : 1 \leq j \leq k\}$ in M_n . Let $z_j = \pi(\{y_j^{(m)}\}) \in C^b(\mathbb{N}, M_n)$, where $\pi : C^b(\mathbb{N}, M_n) \rightarrow C^b(\mathbb{N}, M_n)/C_0(\mathbb{N}, M_n)$ is the quotient map. Then $z_j z_i = z_i z_j$, $i, j = 1, 2, \dots, k$. Let $\Omega = \{(r_1, r_2, \dots, r_k) \in \mathbb{C}^k : |r_j| \leq 1, 1 \leq j \leq k\}$. Define $\varphi : C(\Omega) \rightarrow C^b(\mathbb{N}, M_n)/C_0(\mathbb{N}, M_n)$ by $\varphi(f) = f(z_1, z_2, \dots, z_k)$ for all $f \in C(\Omega)$. It follows from [2] that there exists a unital completely positive linear map $L : C(\Omega) \rightarrow C^b(\mathbb{N}, M_n)$ such that $\pi \circ L = \varphi$. One may write $L(f) = \{L_m(f)\}_{m \in \mathbb{N}}$. One then applies 2.2 to L_m (for all sufficiently large m) to obtain unital homomorphisms $\varphi_m : C(\Omega) \rightarrow M_n$ such that $\lim_{m \rightarrow \infty} \|\varphi_m(f) - L_m(f)\| = 0$ for all $f \in C(\Omega)$. A contradiction would be reached as in the proof of 2.2. \square

Remark 2.4. Note that, in 2.2, δ depends on X , ϵ as well as n . In Corollary 2.3, δ depends on both k and n . In the case that $k = 2$, the theorem for which δ does not depend on n was first proved in [10]. That result is much deeper and was false if $k \geq 3$ (see [4]).

3 Commutative C^* -subalgebras of matrix algebras

Lemma 3.1. *Let X be a compact metric space and let $n \geq 1$ be an integer. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset \mathcal{G} and $\delta > 0$ satisfying the following. Let $\varphi, \psi : C(X) \rightarrow M_n$ be two unital homomorphisms for which*

$$|\text{tr} \circ \varphi(g) - \text{tr} \circ \psi(g)| < \delta \quad \text{for all } g \in \mathcal{G}, \quad (\text{e 3.17})$$

where tr is the normalized tracial state. Then there exists a unitary $u \in M_n$ such that

$$\|\text{ad } u \circ \varphi(f) - \psi(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e 3.18})$$

Proof. Let $\eta > 0$ be such that, for any $x, x' \in X$,

$$|f(x) - f(x')| < \epsilon \quad \text{for all } f \in \mathcal{F},$$

provided $\text{dist}(x, x') < \eta$. Let $\{x_1, x_2, \dots, x_m\}$ be an $\eta/8$ -dense subset of X . For each subset $F \subset \{x_1, x_2, \dots, x_m\}$, define $g_F \in C(X)$ to be a function with $0 \leq g_F(x) \leq 1$ for all $x \in X$, $g_F(x) = 1$ if $x \in O(F, \eta/4)$, and $g_F(x) = 0$, if $x \notin O(F, \eta/2)$.

Let $\delta = 1/2n$ and let $\mathcal{G} = \{g_F : F \subset \{x_1, x_2, \dots, x_m\}\}$. Suppose that φ and ψ satisfy the assumption for the above δ and \mathcal{G} . Let $\{\xi_1, \xi_2, \dots, \xi_n\}$ and $\{\zeta_1, \zeta_2, \dots, \zeta_n\} \subset X$ be such that

$$\varphi(f) = \sum_{i=1}^n f(\xi_i) p_i \quad \text{and} \quad \psi(f) = \sum_{i=1}^n f(\zeta_i) q_i$$

for all $f \in C(X)$, where $\{p_1, p_2, \dots, p_n\}$ and $\{q_1, q_2, \dots, q_n\}$ are two sets of mutually orthogonal rank one projections. Fix a subset $S \subset \{\xi_1, \xi_2, \dots, \xi_n\}$ of k ($1 \leq k \leq n$) elements. For each $\xi_j \in S$, $\xi_j \in B(x_i, \eta/8)$ for some i . Let $F = \{x_i : \text{dist}(x_i, S) < \eta/8\}$. Then

$$k/n \leq \mu_{\text{tro}\varphi}(O(S, \eta/8)) \leq \mu_{\text{tr}} \circ \varphi(O(F, \eta/4)) \quad (\text{e 3.19})$$

$$\leq \text{tr} \circ \varphi(g_F) < \text{tr} \circ \psi(g_F) + \delta \leq \mu_{\text{tro}\psi}(O(F, \eta/2)) + 1/2n. \quad (\text{e 3.20})$$

Note that

$$\mu_{\text{tro}\psi}(O(F, \eta/2)) = k_1/n$$

for some non-negative integers k_1 . Thus, from (e 3.19) and (e 3.20),

$$k/n < k_1/n + 1/2n.$$

It follows that $k \leq k_1$. Thus

$$\mu_{\text{tro}\varphi}(O(S, \eta/8)) \leq \mu_{\text{tro}\psi}(O(F, \eta/2)) \quad (\text{e 3.21})$$

for any $S \subset \{\xi_1, \xi_2, \dots, \xi_n\}$ of k elements. This implies that there is a finite subset $S_1 \subset \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ of at least k elements such that $S_1 \subset O(F, \eta/2)$. It follows that, for each $\zeta_j \in S_1$, there is $x_i \in F$ such that $\xi_i \in O(x_i, \eta/2)$. By the definition of F , there exists $\xi_l \in S$ such that $\zeta_j \in O(\xi_l, 5\eta/8)$.

In other words, $O(S, 5\eta/8)$ contains at least k elements of $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$. Therefore, by the Marriage Law (see [5]), there exists a permutation $\gamma : (1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$ such that

$$\text{dist}(\xi_i, \zeta_{\gamma(i)}) < \eta, \quad i = 1, 2, \dots, n.$$

Let U be the unitary such that $U^* p_i U = q_{\gamma(i)}$, $i = 1, 2, \dots, n$. Then

$$U^* \varphi(f) U = \sum_{i=1}^n f(\xi_i) q_{\gamma(i)} \quad \text{for all } f \in C(X).$$

It follows that

$$\|U^* \varphi(f) U - \psi(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

□

Lemma 3.2. *Let X be a compact metric space which is locally path connected and let $n \geq 1$. Then, for any $\epsilon > 0$, $\epsilon_1 > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset \mathcal{G} and $\delta > 0$ satisfying the following. Let $\varphi, \psi : C(X) \rightarrow M_n$ be two unital homomorphisms for which*

$$|\text{tr} \circ \varphi(g) - \text{tr} \circ \psi(g)| < \delta \quad \text{for all } g \in \mathcal{G}, \quad (\text{e 3.22})$$

where tr is the normalized tracial state. Then there exists a unital homomorphism $\Phi : C(X) \rightarrow C([0, 1], M_n)$ such that

$$\Phi(f)(0) = \varphi(f) \quad \text{and} \quad \|\Phi(f)(t) - \varphi(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and } t \in [0, 1], \quad (\text{e 3.23})$$

and there exists a unitary $u \in M_n$ such that

$$\text{ad } u \circ \Phi(f)(1) = \psi(f) \text{ for all } f \in C(X). \quad (\text{e 3.24})$$

Moreover, there are continuous maps $\alpha_i : [0, 1] \rightarrow X$ ($i = 1, 2, \dots, n$) and mutually orthogonal rank one projections $\{p_1, p_2, \dots, p_n\} \subset M_n$ such that

$$\Phi(f) = \sum_{i=1}^n f(\alpha_i) p_i \text{ for all } f \in C(X) \text{ and} \quad (\text{e 3.25})$$

$$\text{dist}(\alpha_i(t), \alpha_i(0)) < \eta \text{ for all } t \in [0, 1], \quad i = 1, 2, \dots, n. \quad (\text{e 3.26})$$

Proof. This follows from the proof of 3.1. At the beginning of the proof of 3.1, we may further require that there is $\eta_1 > 0$ such that each open ball B_{η_1} with radius η_1 is contained in a path connected neighborhood $Z_\eta \subset B_\eta$. Let δ be as in the proof of 3.1 and choose \mathcal{G} as in the proof of 3.1 (but for η_1 instead of η). Write $\varphi(f) = \sum_{i=1}^n f(\xi_i) p_i$ and $\psi(f) = \sum_{i=1}^n f(\zeta_i) q_i$ for all $f \in C(X)$ as in the proof of 3.1. The proof of 3.1 provides a permutation $\gamma : (1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$ such that

$$\text{dist}(\xi_i, \zeta_{\gamma(i)}) < \eta_1. \quad (\text{e 3.27})$$

Since each open ball $B(\xi_i, \eta_1) \subset Z_{i, \eta_2}$ for some path connected neighborhood contained in $B(x_i, \eta_2)$, where $\eta_2 = \min\{\epsilon_1, \eta\}$, there exist a continuous path $\alpha_i : [0, 1] \rightarrow Z_{i, \eta_2} \subset B(\xi_i, \eta_2)$ such that

$$\alpha_i(0) = \xi_i \text{ and } \alpha_i(1) = \zeta_i, \quad i = 1, 2, \dots, n.$$

Define $\Phi : C(X) \rightarrow C([0, 1], M_n)$ by $\Phi(f)(t) = \sum_{i=1}^n f(\alpha_i(t)) p_i$ for all $f \in C(X)$ and $t \in [0, 1]$. Since $\alpha_i(t) \in Z_{i, \eta_2} \subset B(\xi_i, \eta_2)$, one estimates that

$$\|\Phi(f)(t) - \varphi(f)\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 3.28})$$

and $t \in [0, 1]$. Moreover,

$$\Phi(f)(1) = \sum_{i=1}^n f(\zeta_i) p_i \text{ for all } f \in C(X). \quad (\text{e 3.29})$$

Since $\{q_i : i = 1, 2, \dots, n\}$ and $\{p_i : i = 1, 2, \dots, n\}$ are assumed to be two sets of mutually orthogonal rank one projections, as in the proof of 3.1, one obtains a unitary $u \in M_n$ such that

$$u^* \Phi(f)(1) u = \varphi(f) \text{ for all } f \in C(X). \quad (\text{e 3.30})$$

□

Lemma 3.3. *Let $\epsilon > 0$, $n \geq 1$ be an integer and $M > 0$. There exists $\delta > 0$ satisfying the following: For any finite subset $\mathcal{F} \subset M_n$ with $\|a\| \leq M$ for all $a \in \mathcal{F}$ and a unitary $u \in M_n$ such that*

$$\|ua - au\| < \delta \text{ for all } a \in \mathcal{F},$$

there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset M_n$ with $u(0) = u$ and $u(1) = 1$ such that

$$\|u(t)a - au(t)\| < \epsilon \text{ for all } a \in \mathcal{F}.$$

Moreover,

$$\text{Length}(\{u(t)\}) \leq 2\pi.$$

Proof. Note that $\mathbb{T} \setminus \text{sp}(u)$ contains an arc with length at least $2\pi/n$. Thus the lemma follows immediately from Lemma 2.6.11 of [11].

□

4 Commutative C^* -subalgebras in matrix algebras over one dimension spaces

Lemma 4.1. *Let X be a path connected finite CW complex, let $C = C(X)$ and let $A = C([0, 1], M_n)$. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a finite subset $\mathcal{G} \subset C$ and $\delta > 0$ satisfying the following: Let $\varphi, \psi : C \rightarrow A$ be two unital homomorphisms such that*

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \text{ and for all } \tau \in T(A). \quad (\text{e 4.31})$$

Then there exists a unitary $U \in A$ such that

$$\|\text{ad } U \circ \varphi(f) - \psi(f)\| < \epsilon \quad (\text{e 4.32})$$

for all $f \in \mathcal{F}$.

Moreover, if, in addition, $\varphi(f)(0) = \psi(f)(0)$, or $\varphi(f)(0) = \psi(f)(0)$ and $\varphi(f)(1) = \psi(f)(1)$ for all $f \in C(X)$, then one may assume that $U(0) = 1_{M_n}$, or $U(0) = U(1) = 1$, respectively.

Proof. Without loss of generality, we may assume that \mathcal{F} is in the unit ball of $C(X)$. Put $d = 2\pi/n$. Let $\delta_0 > 0$ (in place of δ) be as required by Lemma 2.6.11 of [11] for $\epsilon/4$. Let $0 < \delta_1 < 1/2n$ (in place of δ) and \mathcal{G}_1 (in place of \mathcal{G}) associated with \mathcal{F} and $\min\{\epsilon/8, \delta_0/4\}$ (in place of ϵ) as required by 3.2.

Put $\epsilon_1 = \min\{\epsilon/16, \delta_1/4, \delta_0/4\}$. There exists $\eta > 0$ such that

$$|g(t) - g(t')| < \epsilon_1/2 \text{ for all } g \in \varphi(\mathcal{G}_0 \cup \mathcal{F}) \cup \psi(\mathcal{G}_0 \cup \mathcal{F}).$$

provided that $|t - t'| < \eta_1$. Choose a partition of the interval:

$$0 = t_0 < t_1 < \dots < t_N = 1$$

such that $|t_i - t_{i-1}| < \eta_1$, $i = 1, 2, \dots, N$. Then

$$\|\varphi(f)(t_i) - \varphi(f)(t_{i-1})\| < \epsilon_1 \text{ and } \|\psi(f)(t_i) - \psi(f)(t_{i-1})\| < \epsilon_1 \quad (\text{e 4.33})$$

for all $f \in \mathcal{G}_0 \cup \mathcal{F}$, $i = 1, 2, \dots, N$. There are unitaries $U_i \in M_n$ and $\{x_{i,j}\}_{j=1}^n$ such that

$$\varphi(f)(t_i) = U_i^* \begin{pmatrix} f(x_{i,1}) & & \\ & \ddots & \\ & & f(x_{i,n}) \end{pmatrix} U_i,$$

$i = 0, 1, 2, \dots, N$. It follows from 3.1 that there exists, for each i , a unitary $W_i \in M_n$ such that

$$\|W_i^* \varphi(f)(t_i) W_i - \psi(f)(t_i)\| < \min\{\epsilon/8, \delta_0/4\} \quad (\text{e 4.34})$$

for all $f \in \mathcal{G}_0 \cup \mathcal{F}$. We estimate, from (e 4.33), that

$$\|\varphi(f)(t_i) - W_{i-1} W_i^* \varphi(f)(t_i) W_i W_{i-1}^*\| < \epsilon_1 + 2 \min\{\epsilon/8, \delta_0/4\} \text{ for all } f \in \mathcal{G}_0 \cup \mathcal{F}, \quad (\text{e 4.35})$$

$i = 1, 2, \dots, N$. By the choice of ϵ_1 and applying Lemma 2.6.11 of [11], we obtain $h_i \in (M_n)_{s.a.}$ such that $W_i W_{i-1}^* = \exp(\sqrt{-1} h_i)$,

$$\|h_i \varphi(f)(t_i) - \varphi(f)(t_i) h_i\| < \epsilon/4 \text{ and} \quad (\text{e 4.36})$$

$$\|\exp(\sqrt{-1} t h_i) \varphi(f)(t_i) - \varphi(f)(t_i) \exp(\sqrt{-1} t h_i)\| < \epsilon/4 \quad (\text{e 4.37})$$

for all $f \in \mathcal{F}$ and $t \in [0, 1]$, $i = 1, 2, \dots, N$. Thus one obtains a continuous path of unitaries $\{Z(t) : t \in [t_{i-1}, t_i]\}$ with $Z(t_{i-1}) = 1_{M_n}$, $Z(t_i) = W_i W_{i-1}^*$, and

$$\|Z^*(t)\varphi(f)(t_i)Z(t) - \varphi(f)(t_i)\| < \epsilon/4 \text{ for all } f \in \mathcal{F}, \quad (\text{e 4.38})$$

for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \dots, N$. One can also apply 3.3 to obtain the path $Z(t)$. Define $W(t) = Z(t)W_{i-1}$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \dots, N$. Note that $W \in C([0, 1], M_n)$. Moreover, for $t \in [t_{i-1}, t_i]$,

$$\|W^*(t)\varphi(f)(t)W(t) - \psi(f)(t)\| \leq \|W^*(t)\varphi(f)(t)W(t) - W^*(t)\varphi(f)(t_i)W(t)\| \quad (\text{e 4.39})$$

$$+ \|W^*(t)\varphi(f)(t_i)W(t) - \psi(f)(t_i)\| + \|\psi(f)(t_i) - \psi(f)(t)\| \quad (\text{e 4.40})$$

$$< \epsilon_1 + \|W_{i-1}^* Z(t)^* \varphi(f)(t_i) Z(t) W_{i-1} - \psi(f)(t_i)\| + \epsilon_1 \quad (\text{e 4.41})$$

$$< \epsilon_1 + \epsilon/4 + \|W_{i-1}^* \varphi(f)(t_i) W_{i-1} - \psi(f)(t_i)\| + \epsilon_1 \quad (\text{e 4.42})$$

$$< 2\epsilon_1 + \epsilon/4 + \epsilon_1 + \|W_{i-1}^* \varphi(f)(t_{i-1}) W_{i-1} - \psi(f)(t_{i-1})\| + \epsilon_1 \quad (\text{e 4.43})$$

$$< 4\epsilon_1 + \epsilon/4 + \epsilon/4 < \epsilon \quad (\text{e 4.44})$$

for all $f \in \mathcal{F}$, $i = 1, 2, \dots, N$.

Finally, if $\varphi(f)(0) = \psi(f)(0)$ for all $f \in C(X)$, we choose $W_0 = 1_{M_n}$. The above proof shows that $W(0) = 1_{M_n}$. Moreover, if $\varphi(f)(1) = \psi(f)(1)$, we choose $M_N = 1_{M_n}$. The above proof also shows that $W(1) = 1_{M_n}$. \square

Lemma 4.2. *Let X be a connected CW complex and let $n \geq 1$. Fix a unital homomorphism $h_0 : C(X) \rightarrow M_n$ given by*

$$h_0(f) = \sum_{i=1}^m f(\xi_i) e_i \text{ for all } f \in C(X),$$

where $\{\xi_1, \xi_2, \dots, \xi_m\}$ ($m \leq n$) is a subset of m distinct points in X and $\{e_1, e_2, \dots, e_m\}$ is a set of mutually orthogonal non-zero projections. For any $\epsilon > 0$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: Suppose that Y is a connected compact metric space, $\varphi : C(X) \rightarrow C(Y, M_n)$ is a unital homomorphism and $y_0 \in Y$ for which

$$\varphi(f)(y_0) = h_0(f) \text{ for all } f \in C(X), \quad (\text{e 4.45})$$

$$\|\varphi(g)(y) - \varphi(g)(y_0)\| < \delta \text{ for all } g \in \mathcal{G}, y \in Y \quad (\text{e 4.46})$$

and there are continuous maps $x_i : Y \rightarrow X$ ($i = 1, 2, \dots, n$) and mutually orthogonal rank one projections $\{q_1, q_2, \dots, q_n\} \subset C(Y, M_n)$ such that

$$\varphi(f) = \sum_{j=1}^n f(x_j) q_j \text{ for all } f \in C(X). \quad (\text{e 4.47})$$

Then, there is a partition $\{S_1, S_2, \dots, S_m\}$ of $\{1, 2, \dots, n\}$ such that $\xi_i = x_j(y_0)$ for some $j \in S_i$, $\text{dist}(\xi_i, x_j(y)) < \epsilon$ for all $j \in S_i$, $\lim_{y \rightarrow y_0} x_j(y) = \xi_i$ for all $j \in S_i$,

$$\|e_i - \sum_{j \in S_i} q_j(y)\| < \epsilon \text{ for all } y \in Y \text{ and } \lim_{y \rightarrow y_0} \sum_{j \in S_i} q_j(y) = e_i, \quad (\text{e 4.48})$$

$i = 1, 2, \dots, m$.

Proof. Let

$$\eta_0 = \min\{\text{dist}(\xi_i, \xi_j) : i \neq j, i, j \in \{1, 2, \dots, m\}\}.$$

Fix any $\eta > 0$ for which $\eta < \min\{\epsilon, \eta_0/2\}$. From the proof of 3.1, there is $\delta_1 > 0$ and a finite subset $\mathcal{G}_1 \subset C(X)$ satisfying the following: if

$$\|\varphi'(g) - h_0(g)\| < \delta_1 \text{ for all } g \in \mathcal{G}$$

for any unital homomorphism $\varphi' : C(X) \rightarrow M_n$, then φ' may be written as

$$\varphi'(f) = \sum_{i=1}^n f(x'_i) p'_i \text{ for all } f \in C(X),$$

where $\{x'_1, x'_2, \dots, x'_n\} \subset X$ and $\{p'_1, p'_2, \dots, p'_n\}$ is a set of mutually orthogonal rank one projections and $\text{dist}(\xi_i, x'_j) < \eta/2$, for $j \in S_i$, where $\{S_1, S_2, \dots, S_m\}$ is a partition of $\{1, 2, \dots, n\}$.

Choose $f_i \in C(X)_+$ with $f_i(x) \leq 1$ such that $f_i(x) = 1$ if $\text{dist}(x, \xi_i) < \eta/2$ and $f_i(x) = 0$ if $\text{dist}(x, \xi_i) \geq \eta$. Put $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{F} \cup \{1, f_i : i = 1, 2, \dots, m\}$. Let $\delta = \min\{\epsilon/4, \eta/2, \delta_1/2\}$. Now if $\varphi : C(X) \rightarrow C(Y, M_n)$ is a unital homomorphism which satisfies (e 4.45) and (e 4.46) for the above δ and \mathcal{G} .

For each $y \in Y$, there is a partition $S_1(y), S_2(y), \dots, S_m(y)$ of $\{1, 2, \dots, n\}$ such that

$$\varphi(f)(y) = \sum_{i=1}^m \left(\sum_{j \in S_i(y)} f(x_j(y)) q_j(y) \right) \text{ for all } f \in C(X), \quad (\text{e 4.49})$$

and $\text{dist}(\xi_i, x_j) < \eta/2$ for all $j \in S_i(y)$, $i = 1, 2, \dots, m$. Since $\eta < \eta_0/2$, $i \in S_i(y)$ for $i = 1, 2, \dots, m$. Suppose that, for some $j \in \{1, 2, \dots, n\}$, there are $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $j \in S_i(y_1)$ but $j \notin S_i(y_2)$. Then $j \in S_{i'}(y_2)$ and $i' \neq i$.

Thus

$$\text{dist}(\xi_i, x_j(y_1)) < \eta/2 \text{ and } \text{dist}(\xi_{i'}, x_j(y_2)) < \eta/2.$$

Note that $\text{dist}(\xi_i, \xi_{i'}) \geq \eta_0$. Hence

$$\text{dist}(\xi_i, x_j(y_2)) \geq \eta_0 - \eta/2 \geq \eta_0 - \eta_0/4 = 3\eta_0/4.$$

Since Y is path connected, there should be a point $y_3 \in Y$ such that

$$\text{dist}(x_j(y_3), \xi_i) = \eta_0/2 > \eta/2.$$

But $\text{dist}(x_j(y_3), \xi_l) < \eta/2$ for some $l \neq i$. However,

$$\text{dist}(\xi_i, \xi_l) \leq \text{dist}(\xi_i, x_j(y_3)) + \text{dist}(x_j(y_3), \xi_l) < \eta_0/2 + \eta/2 < 3\eta_0/4$$

A contradiction. Therefore, if $j \in S_i(y)$, then, for all $y \in Y$, $j \in S_i(y)$. This implies that $S_i(y) = S_i$ is independent of y . The above also implies that $\xi_i = x_j(y_0)$ for some $j \in S_i$. The continuity of $x_j(y)$ also forces

$$\lim_{y \rightarrow y_0} x_j(y) = \xi_i$$

for all $j \in S_i$, $i = 1, 2, \dots, m$.

To finish the proof, one notes that, for each $y \in Y$,

$$\varphi(f_i)(y_0) = e_i \text{ and } \varphi(f_i)(y) = \sum_{j \in S_i} q_j(y), \quad i = 1, 2, \dots, m. \quad (\text{e 4.50})$$

Therefore

$$\|e_i - \sum_{j \in S_i} q_j\| < \eta, \quad i = 1, 2, \dots, m. \quad (\text{e 4.51})$$

Furthermore, by (e 4.50),

$$\lim_{y \rightarrow y_0} \sum_{j \in S_i} q_j(y) = \lim_{y \rightarrow y_0} \varphi(f_i)(y) = \varphi(f_i)(y_0) = e_i, \quad i = 1, 2, \dots, m. \quad (\text{e 4.52})$$

□

Definition 4.3. Let X be a compact metric space, let $n \geq 1$ be an integer and let $t \in X$ be a point. In what follows, denote by $\pi_t : C(X, M_n) \rightarrow M_n$ the point-evaluation homomorphism defined by $\pi_t(f) = f(t)$ for all $f \in C(X)$.

Lemma 4.4. Let X be a locally path connected compact metric space and let $n \geq 1$ be an integer. Then, for any $\epsilon > 0$, $\eta > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: if $\varphi_1, \varphi_2 : C(X) \rightarrow M_n$ are two unital homomorphisms for which

$$\|\varphi_1(g) - \varphi_2(g)\| < \delta \quad \text{for all } g \in \mathcal{G}, \quad (\text{e 4.53})$$

then there is a unital homomorphism $\Phi : C(X) \rightarrow C([0, 1], M_n)$ and there are continuous maps $\alpha_i : [0, 1] \rightarrow X$ ($1 \leq i \leq n$) and mutually orthogonal rank one projections $\{p_1, p_2, \dots, p_n\} \subset C([0, 1], M_n)$ such that $\pi_0 \circ \Phi = \varphi_1$, $\pi_1 \circ \Phi = \varphi_2$,

$$\Phi(f) = \sum_{i=1}^n g(\alpha_i) p_i \quad \text{for all } g \in C(X) \quad \text{and} \quad (\text{e 4.54})$$

$$\|\pi_t \circ \Phi(f) - \varphi_1(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e 4.55})$$

Moreover,

$$\text{dist}(\alpha_i(t), \alpha_i(0)) < \epsilon \quad \text{for all } f \in C(X) \quad \text{and for all } t \in [0, 1]. \quad (\text{e 4.56})$$

Proof. Let $M = \sup\{\|f\| : f \in \mathcal{F}\}$. Let $\delta_1 > 0$ (in place of δ) be as in 3.3 associated with $\epsilon/4$ (in place of ϵ) and M . Let $\delta_2 = \min\{\epsilon/4, \delta_1/2\}$. Let $\delta > 0$ and a finite $\mathcal{G} \subset C(X)$ be as in 3.2 associated with δ_2 (in place of ϵ), $\eta > 0$ (in place of ϵ_1) and \mathcal{F} .

Now suppose that (e 4.53) holds for the above \mathcal{G} and δ . It follows from 3.2 that there exist continuous maps $\alpha'_i : [0, 1/2] \rightarrow X$ ($i = 1, 2, \dots, n$) and mutually orthogonal rank one projections $\{e_1, e_2, \dots, e_n\} \subset M_n$ such that

$$\sum_{i=1}^n f(\alpha'_i(0)) e_i = \varphi_1(f) \quad \text{and} \quad \left\| \sum_{i=1}^n f(\alpha'_i(t)) e_i - \varphi_1(f) \right\| < \epsilon_1 \quad \text{for all } f \in \mathcal{F} \quad \text{and} \quad (\text{e 4.57})$$

for all $t \in [0, 1/2]$ and there exists a unitary $u \in M_n$ such that

$$\text{ad } u \circ \sum_{i=1}^n f(\alpha'_i(1/2)) e_i = \varphi_2(f) \quad \text{for all } f \in C(X). \quad (\text{e 4.58})$$

In particular,

$$\left\| u \left(\sum_{i=1}^n f(\alpha'_i(1/2)) e_i \right) - \left(\sum_{i=1}^n f(\alpha'_i(1/2)) e_i \right) u \right\| < 2\epsilon_1 \quad (\text{e 4.59})$$

for all $f \in \mathcal{F}$. By applying 3.3, there exists a continuous path of unitaries $\{u(t) : t \in [1/2, 1]\} \subset M_n$ such that

$$u(1/2) = 1, \quad u(1) = u \quad \text{and} \quad \left\| u(t) \left(\sum_{i=1}^n f(\alpha'_i(1/2)) e_i \right) - \left(\sum_{i=1}^n f(\alpha'_i(1/2)) e_i \right) u \right\| < \epsilon/4 \quad (\text{e 4.60})$$

for all $f \in \mathcal{F}$. Define $\alpha_i(t) = \alpha'_i(t)$ if $t \in [0, 1/2]$ and $\alpha_i(t) = \alpha'_i(1/2)$ if $t \in (1/2, 1]$, $i = 1, 2, \dots, n$. Define $p_i(t) = e_i$ if $t \in (1/2, 1]$ and $p_i(t) = u(t)^* e_i u(t)$ if $t \in [0, 1/2]$. Then

$$\sum_{i=1}^n f(\alpha_i(0))p_i(0) = \varphi_1(f), \quad \sum_{i=1}^n f(\alpha_i(1))p_i(1) = \varphi_2(f) \quad \text{for all } f \in C(X) \quad \text{and (e 4.61)}$$

$$\left\| \sum_{i=1}^n f(\alpha_i(t))p_i(t) - \varphi_1(f) \right\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e 4.62})$$

□

Lemma 4.5. *Let X be a connected finite CW complex and let $n \geq 1$. Let Y be a finite CW complex of dimension 1 and let $\varphi : C(X) \rightarrow C(Y, M_n)$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist mutually orthogonal rank one projections $p_1, p_2, \dots, p_n \subset C(Y, M_n)$ and continuous maps $\alpha_i : Y \rightarrow X$ such that*

$$\left\| \varphi(f) - \sum_{i=1}^n f(\alpha_i)p_i \right\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \quad (\text{e 4.63})$$

Moreover, if $\{y_1, y_2, \dots, y_L\}$ is fixed, then one can also require that

$$\varphi(f)(y_l) = \sum_{i=1}^n f(\alpha_i(y_l))p_i(y_l) \quad \text{for all } f \in C(X), \quad (\text{e 4.64})$$

$l = 1, 2, \dots, L$.

Proof. The proof of the first part follows that of 4.1 [13]. Since $\dim X = 1$, one can choose a finite subset $\{\zeta_1, \zeta_2, \dots, \zeta_K\} \subset X$ which is ordered in the way so that ζ_{j-1} and ζ_j are connected by a path which is homeomorphic to a line segment and X is the union of these paths (line segments). Without loss of generality, one may assume that these paths are line segments and will be written as $[\zeta_{j-1}, \zeta_j]$, $j = 1, 2, \dots, K$. Furthermore, one may assume that $\{y_1, y_2, \dots, y_L\} \subset \{\zeta_1, \zeta_2, \dots, \zeta_K\}$.

By adding sufficiently many points to $\{\zeta_1, \zeta_2, \dots, \zeta_K\}$, one may also assume that

$$|f(t) - f(\zeta_{j-1})| < \epsilon/3 \quad \text{for all } f \in \mathcal{F} \quad (\text{e 4.65})$$

and all $t \in [\zeta_{j-1}, \zeta_j]$, $j = 1, 2, \dots, K$.

For each j , by applying 4.4, there are continuous maps $\alpha_{i,j} : [\zeta_{j-1}, \zeta_j] \rightarrow X$ ($i = 1, 2, \dots, n$) and mutually orthogonal rank one projections $\{p_{1,j}, p_{2,j}, \dots, p_{n,j}\} \subset C([\zeta_{j-1}, \zeta_j], M_n)$ such that

$$\sum_{i=1}^n f(\alpha_{i,j}(\zeta_{j-1}))p_i(\zeta_{j-1}) = \pi_{\zeta_{j-1}} \circ \varphi, \quad \sum_{i=1}^n f(\alpha_{i,j}(\zeta_j))p_i(\zeta_j) = \pi_{\zeta_j} \circ \varphi \quad \text{and} \quad (\text{e 4.66})$$

$$\left\| \sum_{i=1}^n f(\alpha_{i,j}(t))p_i(t) - \pi_{\zeta_{j-1}} \circ \varphi(f) \right\| < \epsilon/3 \quad \text{for all } f \in \mathcal{F}, \quad (\text{e 4.67})$$

$j = 1, 2, \dots, K$. Define $\alpha_i : Y \rightarrow X$ such that $\alpha_i(t) = \alpha_{i,j}(t)$ for $t \in [\zeta_{j-1}, \zeta_j]$, and define $p_i(t) = p_{i,j}(t)$ for $t \in [\zeta_{j-1}, \zeta_j]$, $j = 1, 2, \dots, K$.

Thus

$$\left\| \varphi(f) - \sum_{i=1}^n f(\alpha_i)p_i \right\| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

□

5 Approximate homomorphisms and The Basic Homotopy Lemma

Lemma 5.1. *Let X be a locally path connected compact metric space and let $n \geq 1$ be an integer. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: for any unital map $\varphi : C(X) \rightarrow C([0, 1], M_n)$ with $\|\varphi(f)\| \leq M$ for all $\|f\| \leq 1$ such that*

$$\|\varphi(\lambda_1 x + \lambda_2 y) - (\lambda_1 \varphi(x) + \lambda_2 \varphi(y))\| < \delta, \quad (\text{e 5.68})$$

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| < \delta \text{ and } \|\varphi(x^*) - \varphi(x)^*\| < \delta \quad (\text{e 5.69})$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_i| \leq 1$ ($i = 1, 2$) and $x, y \in \mathcal{G}$, there exists a unital homomorphism $\psi : C(X) \rightarrow C([0, 1], M_n)$ such that

$$\|\varphi(f) - \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 5.70})$$

If, moreover, $\pi_0 \circ \varphi$ is a unital homomorphism, or both $\pi_0 \circ \varphi$ and $\pi_1 \circ \varphi$ are unital homomorphisms, then ψ can be so chosen that $\pi_0 \circ \psi = \pi_0 \circ \varphi$ (or $\pi_0 \circ \psi = \pi_1 \circ \varphi$ and $\pi_1 \circ \psi = \pi_1 \circ \varphi$).

Proof. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. Without loss of generality, we may assume that $\|f\| \leq 1$ for all $f \in \mathcal{F}$. Let $\delta_1 > 0$ (in place of δ) and \mathcal{G}_1 (in place of \mathcal{G}) associated with $\epsilon/3$, \mathcal{F} and n required by 4.4. One may assume that $\mathcal{F} \subset \mathcal{G}_1$ and $\delta_1 < \epsilon/2$.

Let $\delta > 0$ and $\mathcal{G} \subset C(X)$ be a finite subset associated with $\delta_1/3$ (in place of ϵ) and \mathcal{G}_1 (in place of \mathcal{F} and M as required by 2.2. One may assume that $\mathcal{G}_1 \subset \mathcal{G}$. Suppose that φ satisfies (e 5.68) and (e 5.69) for the above δ and \mathcal{G} . Let $\eta > 0$ such that

$$|f(t) - f(t')| < \delta_1/3 \text{ for all } f \in \mathcal{G} \quad (\text{e 5.71})$$

if $|t - t'| < \eta$. Let

$$0 = t_0 < t_1 < \dots < t_N = 1$$

be a partition such that $|t_i - t_{i-1}| < \eta$, $i = 1, 2, \dots, N$. It follows from 2.2 that, for each i , there exists a unital homomorphism $\psi_i : C(X) \rightarrow M_n$ such that

$$\|\varphi(f)(t_i) - \psi_i(f)\| < \delta_1/3 \text{ for all } f \in \mathcal{F} \cup \mathcal{G}_1. \quad (\text{e 5.72})$$

One estimates that

$$\|\psi_i(f) - \psi_{i-1}(f)\| \leq \|\psi_i(f) - \varphi(f)(t_i)\| \quad (\text{e 5.73})$$

$$+ \|\varphi(f)(t_i) - \varphi(f)(t_{i-1})\| + \|\varphi(f)(t_{i-1}) - \psi_{i-1}(f)\| \quad (\text{e 5.74})$$

$$< \delta_1/3 + \delta_1/3 + \delta_1/3 = \delta_1 \quad (\text{e 5.75})$$

for all $f \in \mathcal{F} \cup \mathcal{G}_1$, $i = 1, 2, \dots, N$. It follows from 4.4 that, for each i , there is a unital homomorphism $\Phi_i : C(X) \rightarrow C([t_{i-1}, t_i], M_n)$ such that

$$\pi_{t_{i-1}} \circ \Phi_i = \psi_{i-1}, \quad \pi_{t_i} \circ \Phi_i = \psi_i \text{ and} \quad (\text{e 5.76})$$

$$\|\pi_t \circ \Phi_i(f) - \psi_{i-1}(f)\| < \epsilon/3 \quad (\text{e 5.77})$$

for all $t \in [t_{i-1}, t_i]$, $i = 1, 2, \dots, N$. Note that if $\pi_0 \circ \varphi$ (or $\pi_1 \circ \varphi$) is a unital homomorphism, then one can require that $\pi_0 \circ \Phi = \pi_0 \circ \varphi$ (or $\pi_1 \circ \Phi = \pi_1 \circ \varphi$).

Define $\Phi : C(X) \rightarrow C([0, 1], M_n)$ by $\pi_t \circ \Phi = \pi_t \circ \Phi_i$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \dots, N$. It is easy to see that Φ meets the requirements. □

Lemma 5.2. *Let X be a compact metric space and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following:*

Suppose that $\varphi : C(X) \rightarrow C([0, 1], M_n)$ is a unital homomorphism and $u \in C([0, 1], M_n)$ such that

$$\|u\varphi(g) - \varphi(g)u\| < \delta \text{ for all } g \in \mathcal{G}. \quad (\text{e 5.78})$$

Then there exists a continuous path of unitaries $\{U(s) : s \in [0, 1]\}$ in $C([0, 1], M_n)$ with $U(0) = u$ and $U(1) = 1$ such that

$$\|U(s)\varphi(f) - \varphi(f)U(s)\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 5.79})$$

and for all $s \in [0, 1]$. Moreover, if

$$u(0)(\pi_0 \circ \varphi(f)) = (\pi_0 \circ \varphi(f))u(0) \text{ (and } u(1)(\pi_1 \circ \varphi(f)) = (\pi_1 \circ \varphi(f))u(1)) \quad (\text{e 5.80})$$

for all $f \in C(X)$, then one can choose U so that

$$U(s)(0)(\pi_0 \circ \varphi(f)) = (\pi_0 \circ \varphi(f))U(s)(0) \text{ (and } U(s)(1)(\pi_1 \circ \varphi(f)) = (\pi_1 \circ \varphi(f))U(s)(1)) \quad (\text{e 5.81})$$

for all $f \in C(X)$.

Proof. Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Put $Y = X \times \mathbb{T}$. We identify $C(Y)$ with $C(X) \otimes C(\mathbb{T})$. Denote by $\mathcal{F}_1 = \{f \otimes 1, 1 \otimes z : f \in \mathcal{F}\}$, where $z \in C(\mathbb{T})$ is the identity function on the unit circle. Let $\delta > 0$ and let $\mathcal{G}_1 \subset C(Y)$ (in place of \mathcal{G}) be the finite subset required by 5.1 for $\epsilon/4$ and \mathcal{F}_1 and Y (in place of X).

It is easy to see, by choosing smaller δ , one may assume that $\mathcal{G}_1 = \{f \otimes 1, 1 \otimes z : f \in \mathcal{G}\}$ for some finite subset $\mathcal{G} \subset C(X)$. Assume (e 5.78) holds for δ and \mathcal{G} . It follows from 5.1 that there exists a unital homomorphism $\Phi : C(X) \otimes C(\mathbb{T}) \rightarrow C([0, 1], M_n)$ such that

$$\|\Phi(f \otimes 1) - \varphi(f)\| < \epsilon/4 \text{ and } \|\Phi(1 \otimes z) - u\| < \epsilon/4 \quad (\text{e 5.82})$$

for all $f \in \mathcal{F}$. It follows from 4.5 that there are continuous functions $\alpha_i : [0, 1] \rightarrow X$ and $\beta_i : [0, 1] \rightarrow \mathbb{T}$ such that

$$\|\Phi(f \otimes 1) - \sum_{i=1}^n f(\alpha_i)p_i\| < \epsilon/4 \text{ and } \|\Phi(1 \otimes z) - \sum_{i=1}^n \beta_i p_i\| < \epsilon/4 \quad (\text{e 5.83})$$

for all $f \in \mathcal{F}$, where p_1, p_2, \dots, p_n are rank one projections in $C([0, 1], M_n)$.

Since $\beta_i \in C([0, 1], \mathbb{T})$, there exists a continuous path of unitaries $\{u_i(s) : s \in [1/2, 1]\} \subset C([0, 1], M_n)$ such that

$$u_i(1/2) = \beta_i \text{ and } u_i(1) = 1, \quad i = 1, 2, \dots, n. \quad (\text{e 5.84})$$

Define $U(s) = \sum_{i=1}^n u_i(s)p_i$ for $s \in [1/2, 1]$. Note that, for each $s \in [1/2, 1]$, $U(s) \in C([0, 1], M_n)$. Moreover,

$$U(1/2) = \sum_{i=1}^n \beta_i p_i, \quad u(1) = 1 \text{ and } U(s)(\sum_{i=1}^n f(\alpha_i)p_i) = (\sum_{i=1}^n f(\alpha_i)p_i)U(s) \quad (\text{e 5.85})$$

for all $f \in C(X)$ and $s \in [1/2, 1]$. Thus, by (e 5.82) and (e 5.83),

$$\|U(s)\varphi(f) - \varphi(f)U(s)\| < \epsilon/2 \text{ for all } f \in \mathcal{F} \text{ for all } s \in [1/2, 1]. \quad (\text{e 5.86})$$

Since, by (e 5.82) and (e 5.83),

$$\|1_{M_n} - u^*(\sum_{i=1}^n \beta_i p_i)\| < \epsilon/2, \quad (\text{e 5.87})$$

there exists a self-adjoint element $a \in C([0, 1], M_n)$ such that

$$u^*(\sum_{i=1}^n \beta_i p_i) = \exp(ia) \text{ and } \|a\| \leq 2 \arcsin(\epsilon/4). \quad (\text{e 5.88})$$

Define $U(s) = u \exp(i2sa)$ for $s \in [0, 1/2]$. Then $\{U(s) : s \in [0, 1/2]\} \subset C([0, 1], M_n)$ such that

$$U(0) = u, \quad U(1/2) = \sum_{i=1}^n \beta_i(1/2) p_i \text{ and} \quad (\text{e 5.89})$$

$$\|1 - u^*U(s)\| < \epsilon/2 \text{ for all } t \in [0, 1/2] \quad (\text{e 5.90})$$

for all $s \in [0, 1/2]$. Note that $\{U(s) : t \in [0, 1]\} \subset C([0, 1], M_n)$ is a continuous path of unitaries and one estimates that

$$\|\varphi(f)U(s) - U(s)\varphi(f)\| < \epsilon \text{ for all } f \in \mathcal{F} \quad (\text{e 5.91})$$

for all $s \in [0, 1]$.

This proves the first part of the lemma. To prove the last part, by applying the last part of 5.1, one may choose Φ so that

$$\pi_0 \circ \Phi(f \otimes 1) = \pi_0 \circ \varphi(f) \text{ and } \pi_0 \circ \Phi(1 \otimes z) = u(0). \quad (\text{e 5.92})$$

Moreover, by 4.5,

$$\pi_0 \circ \Phi(f \otimes 1) = \sum_{i=1}^n f(\alpha_i(0)) p_i(0) \text{ and } \pi_0 \circ \Phi(1 \otimes z) = \sum_{i=1}^n \beta_i(0) p_i(0). \quad (\text{e 5.93})$$

Therefore,

$$(\pi_0 \circ \varphi(f))U(s)(0) = U(s)(0)(\pi_0 \circ \varphi(f)) \quad (\text{e 5.94})$$

for all $f \in C(X)$ and $s \in [1/2, 1]$. In this case, one also has $a(0) = 0$ in (e 5.88). Therefore $U(s)(0) = u$ for all $s \in [0, 1/2]$. Thus, in fact, (e 5.94) holds for all $s \in [0, 1]$.

One can also make the same arrangement for $t = 1$.

□

6 Approximate diagonalization

Definition 6.1. Let Y be a compact metric space. Recall that Y is a locally absolute retract, if, for any $y \in Y$ and any $\epsilon_1 > 0$, there exist $\epsilon_1 > \epsilon_2 > 0$ and a closed neighborhood Z of y such that $B(y, \epsilon_2) \subset Z \subset B(y, \epsilon_1)$ and Z is an absolute retract.

Lemma 6.2. Let X be a compact metric space which is locally absolutely retract and $n \geq 1$ be an integer. Let \mathbb{T} be the unit circle. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Suppose that $\varphi : C(X) \rightarrow C(\mathbb{T}, M_n)$ is a unital homomorphism satisfying the following:

(1) $\varphi(f) = \sum_{i=1}^n f(\alpha_i) p_i$, where $\alpha_i : \mathbb{T} \rightarrow X$ are continuous maps and $\{p_1, p_2, \dots, p_n\} \subset C(\mathbb{T}, M_n)$ are mutually orthogonal rank one projections;

(2) $\pi_{\sqrt{-1}} \circ \varphi(f) = \sum_{i=1}^m f(x_i)e_i$, where $\{x_1, x_2, \dots, x_m\} \subset X$ are distinct points, $\{e_1, e_2, \dots, e_m\}$ is a set of mutually orthogonal non-zero projections,

(3) There is a partition $\{S_1, S_2, \dots, S_m\}$ of $\{1, 2, \dots, n\}$ such that, for $s \in S_i$,

$$\text{dist}(x_i, \alpha_s(y)) < \eta_1 \text{ for all } y \in X, \quad (\text{e 6.95})$$

$B(x_i, \eta_1) \subset Z_i \subset B(x_i, \eta_2/4)$ and Z_i is a compact subset which is also absolutely retract, $i = 1, 2, \dots, m$, where

$$|f(x) - f(x')| < \min\{\delta_0/2, \epsilon/4\} \text{ for all } f \in \mathcal{G},$$

if $\text{dist}(x, x') < \eta_2$, and where δ_0 (in place of δ) and \mathcal{G} associated with $\epsilon/2$ (in place of ϵ) and \mathcal{F} required by 5.2;

(4)

$$\|\pi_{\sqrt{-1}} \circ \varphi(g) - \pi_t \circ \varphi(g)\| < \delta_0/2 \text{ for all } g \in \mathcal{G} \text{ and } t \in \mathbb{T}. \quad (\text{e 6.96})$$

Then, there exist continuous maps $\gamma_i : D \rightarrow X$ and mutually orthogonal rank one projections $\{q_1, q_2, \dots, q_n\} \subset C(D, M_n)$ such that, for any $t \in \mathbb{T}$ and any $y \in D$,

$$\|\pi_t \circ \varphi(f) - \sum_{i=1}^n f(\gamma_i(y))q_i(y)\| < \epsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 6.97})$$

$$\gamma_i|_{\mathbb{T}} = \alpha_i \text{ and } q_i|_{\mathbb{T}} = p_i, \quad (\text{e 6.98})$$

$i = 1, 2, \dots, n$, where D is the unit disk.

Proof. Let C_+ be the unit upper semi-circle and C_- be the unit lower semi-circle. Let $L_+ = \{1 + a\sqrt{-1} : -1 \leq a \leq 0\}$, $L_- = \{-1 + a\sqrt{-1} : -1 \leq a \leq 0\}$ and let $C'_+ = \{z - \sqrt{-1} : z \in C_+\}$. Denote by Ω the compact subset of the plane with boundary consisting of C_+, C'_+, L_+ and L_- .

For each i ($1 \leq i \leq n$), define $\beta_i : \partial\Omega \rightarrow X$ as follows:

$$\beta_i(y) = \begin{cases} \alpha_i(y), & \text{if } y \in C_+; \\ \alpha_i(-1) & \text{if } y \in L_-; \\ \alpha_i(1) & \text{if } y \in L_+ \text{ and} \\ \alpha_i(e^{-\sqrt{-1}\theta}) & \text{if } y = e^{\sqrt{-1}\theta} - \sqrt{-1}. \end{cases}$$

Note that, by (e 6.95), $\beta_i(y) \in Z_i$. Since Z_i is an absolute retract, there is a continuous map $\bar{\beta}_i : \Omega \rightarrow Z_i$ such that $\bar{\beta}_i|_{\partial\Omega} = \beta_i$, $i = 1, 2, \dots, n$. Define $\Phi_1 : C(X) \rightarrow C(\Omega, M_n)$ by

$$\Phi_1(f)(y) = \sum_{i=1}^n f(\bar{\beta}_i(y))p_i(e^{\sqrt{-1}\theta}) \quad (\text{e 6.99})$$

for $y = e^{\sqrt{-1}\theta} - a\sqrt{-1}$ for some $-1 \leq a \leq 0$ and $0 \leq \theta \leq \pi$, and for all $f \in C(X)$. By (3),

$$\|\Phi(f)(y) - \varphi(f)(\sqrt{-1})\| < \min\{\delta_0/2, \epsilon/4\} \text{ for all } y \in \Omega. \quad (\text{e 6.100})$$

Since p_i has rank one, there exists a unitary $u \in C(C_+, M_n)$ such that

$$u(e^{\sqrt{-1}t})^* p_i(e^{\sqrt{-1}t}) u(e^{\sqrt{-1}t}) = p_i(e^{-\sqrt{-1}t}) \text{ for all } t \in [0, \pi], \quad (\text{e 6.101})$$

$i = 1, 2, \dots, n$. Define $\varphi_1, \varphi_2 : C(X) \rightarrow C(C_+, M_n)$ by

$$\varphi_1(f)(e^{\sqrt{-1}\theta}) = \sum_{i=1}^n f(\alpha_i(e^{-\sqrt{-1}\theta}))p_i(e^{\sqrt{-1}\theta}) \text{ and} \quad (\text{e 6.102})$$

$$\varphi_2(f)(e^{\sqrt{-1}\theta}) = \sum_{i=1}^n f(\alpha_i(e^{-\sqrt{-1}\theta}))p_i(e^{-\sqrt{-1}\theta}) \quad (\text{e 6.103})$$

for all $0 \leq \theta \leq \pi$. Then,

$$u(e^{\sqrt{-1}\theta})^* \varphi_1(f)(e^{\sqrt{-1}\theta}) u(e^{\sqrt{-1}\theta}) = \varphi_2(f)(e^{\sqrt{-1}\theta}) \quad (\text{e 6.104})$$

for all $0 \leq \theta \leq \pi$. By (3),

$$\max_{0 \leq \theta \leq \pi} \|\varphi_1(f)(e^{\sqrt{-1}\theta}) - \varphi(f)(e^{\sqrt{-1}\theta})\| < \min\{\delta_0/2, \epsilon/4\}. \quad (\text{e 6.105})$$

By (4), one has that

$$\|\varphi_2(f) - \varphi(f)\|_{C_+} < \delta_0/2 \text{ for all } f \in \mathcal{G}. \quad (\text{e 6.106})$$

Note that

$$\varphi_1(f)(-1) = \varphi_2(f)(-1) \text{ and } \varphi_1(f)(1) = \varphi_2(f)(1) \quad (\text{e 6.107})$$

for all $f \in C(X)$. It follows from 5.2 that there exists a continuous path of unitaries $\{U_s : s \in [0, 1]\} \subset C(C_+, M_n)$ such that $U_1 = u$, $U_0 = 1_{M_n}$ and, for all $s \in [0, 1]$,

$$\|U_s^* \varphi_1(f) U_s - \varphi_1\|_{C_+} < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e 6.108})$$

Moreover,

$$U_s(-1)^* \varphi_1(f)(-1) U_s(-1) = \varphi_1(f)(-1) \text{ and } U_s(1)^* \varphi_1(f)(1) U_s(1) = \varphi_1(f)(1) \quad (\text{e 6.109})$$

for all $f \in C(X)$.

Let $C'_- = \{z - 2\sqrt{-1} : z \in C_-\}$, $L'_- = \{-1 + b\sqrt{-1} : -2 \leq b \leq -1\}$ and $L'_+ = \{1 + b\sqrt{-1} : -2 \leq b \leq -1\}$. Denote by Ω_1 the compact subset with the boundary consisting of C'_+ , L'_- , L'_+ and C'_- . Define $\Phi_2 : C(X) \rightarrow C(\Omega_1, M_n)$ as follows: If $y = (1-s)(e^{\sqrt{-1}\theta}) - \sqrt{-1} + s(e^{-\sqrt{-1}\theta} - 2\sqrt{-1})$, then define

$$\Phi_2(f)(y) = U_s(e^{\sqrt{-1}\theta})^* \varphi_1(f)(e^{\sqrt{-1}\theta}) U_s(e^{\sqrt{-1}\theta}) \quad (\text{e 6.110})$$

for all $s \in [0, 1]$ and $0 \leq \theta \leq \pi$. Then

$$\Phi_2(f)(y) = \sum_{i=1}^n f(\alpha_i(e^{-\sqrt{-1}\theta})) (U_s^* p_i U_s)(e^{\sqrt{-1}\theta}) \quad (\text{e 6.111})$$

for all $f \in C(X)$, if $y = (1-s)(e^{\sqrt{-1}\theta}) - \sqrt{-1} + s(e^{-\sqrt{-1}\theta} - 2\sqrt{-1})$, $s \in [0, 1]$ and $0 \leq \theta \leq \pi$.

Note that

$$\Phi_2(f)(e^{\sqrt{-1}\theta} - \sqrt{-1}) = \Phi_1(f)(e^{\sqrt{-1}\theta} - \sqrt{-1}), \quad (\text{e 6.112})$$

$$\Phi_2(f)(e^{\sqrt{-1}\theta} - 2\sqrt{-1}) = \varphi(f)(e^{-\sqrt{-1}\theta}) \quad (\text{e 6.113})$$

$$\Phi_2(f)(y) = \varphi(f)(-1) \text{ and } \Phi_2(f)(y') = \varphi(f)(1) \quad (\text{e 6.114})$$

for any $y \in L'_-$, $y' \in L'_+$ and $0 \leq \theta \leq \pi$. Let $\Omega_2 = \Omega \cup \Omega_1$. By (e 6.112), one can define $\Phi : C(X) \rightarrow C(\Omega_2, M_n)$ by

$$\Phi(f)(y) = \Phi_1(f)(y) \text{ if } y \in \Omega \text{ and } \Phi(f)(y) = \Phi_2(f)(y) \text{ if } y \in \Omega_1. \quad (\text{e 6.115})$$

Fix $3/4 < d_0 < 1$. Let

$$S_- = \{-d_0 + (1-d_0)e^{i\theta} : |\theta| \leq \pi\} \text{ and } S_+ = \{d_0 + (1-d_0)e^{i\theta} : |\theta| \leq \pi\}$$

be two small circles with the centers $-d_0$ and d_0 , respectively. Let A be the connected subset containing the origin and bounded by S_- , S_+ , C_+ and C_- . Denote by A^o the interior of A . Then there exists a continuous map $\Gamma : \Omega_2 \rightarrow A$ which is a homeomorphism from the interior of Ω_2 onto A^o , which fixes C_+ , maps C'_- onto C_- such that $\Gamma(e^{\sqrt{-1}\theta} - 2\sqrt{-1}) = e^{\sqrt{-1}\theta}$ and maps $L_- \cup L'_-$ onto S_- and $L_+ \cup L'_+$ onto S_+ .

Now define $\psi : C(X) \rightarrow C(D, M_n)$ by

$$\psi(f)(y) = \begin{cases} \Phi(f(\Gamma^{-1}(y))), & \text{if } y \in A^o; \\ \varphi(f)(y), & \text{if } y \in C_+; \\ \varphi(f)(y), & \text{if } y \in C_-; \\ \varphi(f)(-1), & \text{if } |y + d_0| \leq 1 - d_0; \\ \varphi(f)(1), & \text{if } |y - d_0| \leq 1 - d_0 \end{cases} \quad (\text{e 6.116})$$

for all $f \in C(X)$. That ψ maps $C(X)$ into $C(D, M_n)$ follows from (e 6.99), (e 6.112), (e 6.113) and (e 6.114). By (e 6.99) and (e 6.111), there are continuous maps $\gamma_i : D \rightarrow X$ and mutually orthogonal rank one projections $q_1, q_2, \dots, q_n \in C(D, M_n)$ such that

$$\psi(f)(y) = \sum_{i=1}^n f(\gamma_i(y))q_i(y) \text{ for all } f \in C(X). \quad (\text{e 6.117})$$

Moreover,

$$\gamma_i|_{\mathbb{T}} = \alpha_i \text{ and } q_i|_{\mathbb{T}} = p_i, \quad i = 1, 2, \dots, n \quad (\text{e 6.118})$$

It follows (e 6.100) and (e 6.106)

$$\|\pi_t \circ \varphi(f) - \sum_{i=1}^n f(\gamma_i(y))q_i(y)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 6.119})$$

□

Theorem 6.3. *Let X be a compact metric space which is a locally absolute retract and let $n \geq 1$. Suppose that Y is a compact metric space with $\dim Y \leq 2$ and suppose that $\varphi : C(X) \rightarrow C(Y, M_n)$ is a unital homomorphism. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist continuous maps $\alpha_i : Y \rightarrow X$ ($1 \leq i \leq n$) and mutually orthogonal rank one projections $e_1, e_2, \dots, e_n \in C(Y, M_n)$ such that*

$$\|\varphi(f) - \sum_{i=1}^n f(\alpha_i)e_i\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 6.120})$$

Proof. Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\delta_0 > 0$ (in place of δ) and $\mathcal{G}_1 \subset C(X)$ (in place of \mathcal{G}) be a finite subset associated with $\epsilon/16$ and \mathcal{F} required by 5.2 (for the given n). One may assume that $\mathcal{F} \subset \mathcal{G}_1$. Let $\eta > 0$ be such that

$$|f(x) - f(x')| < \min\{\delta_0/2, \epsilon/16\} \text{ for all } f \in \mathcal{G}_1, \quad (\text{e 6.121})$$

provided that $\text{dist}(x, x') < \eta$. Since X is locally absolute retract and compact, there exists $\eta_1 > 0$ such that, for any $x \in X$, $B(x, \eta_1) \subset Z_x \subset B(x, \eta/2)$, where Z_x is a compact subset which is also an absolute retract. For each $y \in Y$, let $\delta_1(y) > 0$ (in place of δ) and $\mathcal{G}'_2(y) \subset C(X)$ (in place of \mathcal{G}) be a finite subset associated with $\min\{\eta_0/3, \epsilon/16\}$ (in place of ϵ) and $\pi_y \circ \varphi$ required by 4.2.

For each $y \in Y$, let $\delta_2(y) > 0$ (in place of δ) and $\mathcal{G}_2(y) \subset C(X)$ (in place of \mathcal{G}) be a finite subset associated with $\min\{\eta_0/3, \epsilon/6\}$ (in place of ϵ) and $\pi_y \circ \varphi$ (in place of φ_1) required by 4.4. Without loss of generality, to simplify notation, we may assume that $\delta_1(y) \leq \delta_2(y)$ and $\mathcal{G}_1 \cup \mathcal{G}'_2(y) \subset \mathcal{G}_2(y)$.

For each y , there exists $d'(y) > 0$ such that

$$\|\pi_y \circ \varphi(g) - \pi_{y'} \circ \varphi(g)\| < \min\{\delta_0/2, \delta_2(y)/3, \epsilon/16\} \text{ for all } g \in \mathcal{G}_2(y), \quad (\text{e 6.122})$$

provided that $\text{dist}(y, y') < d'(y)$.

Fix $r > 0$. For each $y \in Y$, let $d(y) = d'(y)r$. Now $\cup_{y \in Y} B(y, d(y)/12) \supset Y$. Let $y_1, y_2, \dots, y_K \in Y$ be a finite subset such that $\{B(y_i, d(y_i)/12) : i = 1, 2, \dots, K\}$ covers Y . Moreover, one may assume that the order of the cover ≤ 2 . One builds a simplicial complex as follows: y_1, y_2, \dots, y_N are vertices and 0-simplexes, and $y_{i_1}y_{i_2}$ or $y_{i_1}y_{i_2}y_{i_3}$ form a 1-simplex (or 2-simplex) if and only if

$$B(y_{i_1}, d(y_{i_1})/12) \cap B(y_{i_2}, d(y_{i_2})/12) \neq \emptyset \quad (\text{e 6.123})$$

$$(\text{ and } B(y_{i_1}, d(y_{i_1})/12) \cap B(y_{i_2}, d(y_{i_2})/12) \cap B(y_{i_3}, d(y_{i_3})/12)) \neq \emptyset. \quad (\text{e 6.124})$$

Denote by $\mathcal{S}(r)$ the simplicial complexes constructed this way and by $S(r)$ the underline polyhedra. Moreover, if $y_i y_j$ is a 1-simplex, then

$$\text{dist}(y_i, y_j) < \max\{d(y_i)/6, d(y_j)/6\}. \quad (\text{e 6.125})$$

If y_j is a vertex, then there are points $\alpha_j^{(k)}(y_j) \in X$, $k = 1, 2, \dots, n$, and mutually orthogonal rank one projections $p_1^j, p_2^j, \dots, p_n^j \in M_n$ such that

$$\varphi(f)(y_j) = \sum_{k=1}^n f(\alpha_j^{(k)}(y_j)) p_k^j \text{ for all } f \in C(X). \quad (\text{e 6.126})$$

Denote by $I_{i,j}$ the line segment defined by $y_i y_j$. Therefore, by applying 4.4, there are continuous maps $\alpha_{i,j}^{(k)} : I_{i,j} \rightarrow X$, $k = 1, 2, \dots, n$, and mutually orthogonal rank one projections $p_1^{i,j}, p_2^{i,j}, \dots, p_n^{i,j} \in C(I_{i,j}, M_n)$ such that

$$\sum_{k=1}^n f(\alpha_{i,j}^{(k)}(y_s)) p_k^{i,j}(y_s) = \pi_{y_s} \circ \varphi(f) \text{ for all } f \in C(X), \quad (\text{e 6.127})$$

$$\|\pi_{y_{s'}} \circ \varphi(g) - \sum_{k=1}^n f(\alpha_{i,j}^{(k)}(t)) p_k^{i,j}(t)\| < \min\{\delta_2(y_{s'})/2, \epsilon/16\} \text{ for all } g \in \mathcal{G}_2(y_{s'}), \quad (\text{e 6.128})$$

where $s = i, j$ and $s' = i$, or j if $\max\{d(y_i)/6, d(y_j)/6\} = d(y_i)/6$, or $\max\{d(y_i)/6, d(y_j)/6\} = d(y_j)/6$.

Let $I(r) = \cup I_{i,j}$ be the union of all 0-simplex and 1-simplexes in $S(r)$. One obtains a unital homomorphism $\Phi' : C(X) \rightarrow C(I(r), M_n)$ defined by

$$\pi_t \circ \Phi'(f) = \sum_{k=1}^n f(\alpha_{i,j}^{(k)}(t)) p_k^{i,j}(t) \quad (\text{e 6.129})$$

if $t \in I_{i,j}$, and

$$\pi_{y_j} \circ \Phi' = \pi_{y_j} \circ \varphi. \quad (\text{e 6.130})$$

Define $\alpha'_k : I(r) \rightarrow X$ by $\alpha'_k(t) = \alpha_{i,j}^{(k)}(t)$ if $t \in I_{i,j}$ and define projections p'_i, p'_2, \dots, p'_n in $C(I(r), M_n)$ by $p'_k(t) = p_k^{i,j}(t)$ if $t \in I_{i,j}$. Next one extends $\pi_t \circ \Phi'$ on $S(r)$.

To do this, one assumes that $y_{i_1}y_{i_2}y_{i_3}$ is a 2-simplex. Then

$$\text{dist}(y_{i_j}, y_{i_{j'}}) < d(y_{i_j})/6 \text{ for all } j = 1, 2, 3 \quad (\text{e 6.131})$$

and for one of some $j' \in \{1, 2, 3\}$. Without loss of generality, one may assume that $3 = j'$.

One identifies the 2-polyhedron K_{i_1, i_2, i_3} determined by $y_{i_1}y_{i_2}y_{i_3}$ with the unit disk and identifies y_{i_1} with 1, y_{i_2} with -1 and y_{i_3} with $\sqrt{-1}$. Here the line segments determined by $y_{i_1}y_{i_2}$, $y_{i_1}y_{i_3}$ and $y_{i_2}y_{i_3}$ with the arc with end points -1 and 1 , the arc with end points 1 and $\sqrt{-1}$, and the arc with end points $\sqrt{-1}$ and -1 , respectively.

Let Ψ be the restriction of Φ' on the unit circle \mathbb{T} (with the above mentioned identification). Then it is clear that Ψ satisfies (1), (2) and (4) in 6.2 (by replacing φ by Ψ) for $\epsilon/4$ (in place of ϵ) and \mathcal{F} . By the choice of each $\delta_1(y)$ and by 4.2, (3) is also satisfied (for Ψ). By applying 6.2, and identifying the unit disk D with K_{i_1, i_2, i_3} , one obtains a unital homomorphism $\Phi_{i_1, i_2, i_3} : C(X) \rightarrow C(K_{i_1, i_2, i_3}, M_n)$, continuous maps $\alpha_{i_1, i_2, i_3}^{(k)} : K_{i_1, i_2, i_3} \rightarrow X$ and mutually orthogonal rank one projections $\{p_k^{i_1, i_2, i_3} : k = 1, 2, \dots, n\} \subset C(K_{i_1, i_2, i_3}, M_n)$ such that (where $K_{i_j, i_{j'}}$ is the 1-simplex determined by $y_{i_j}y_{i_{j'}}$)

$$\Phi_{i_1, i_2, i_3}(f) = \sum_{k=1}^n f(\alpha_{i_1, i_2, i_3}^{(k)}) p_k^{i_1, i_2, i_3} \text{ for all } f \in C(X), \quad (\text{e 6.132})$$

$$\Phi_{i_1, i_2, i_3}(f)|_{K_{i_j, i_{j'}}} = \sum_{k=1}^n f(\alpha_{i_j, i_{j'}}^{(k)}) p_k^{i_j, i_{j'}} \text{ for all } f \in C(X) \text{ and} \quad (\text{e 6.133})$$

$$\|\Phi_{i_1, i_2, i_3}(f)(t) - \sum_{k=1}^n f(\alpha_{i_1, i_2, i_3}^{(k)})(s) p_k^{i_1, i_2, i_3}(s)\| < \epsilon/2 \quad (\text{e 6.134})$$

for all t in the boundary of K_{i_1, i_2, i_3} , $s \in K_{i_1, i_2, i_3}$ and for all $f \in \mathcal{F}$. Define $\alpha_k : Y \rightarrow S(r)$ by $\alpha_k(y_j) = y_j$, $\alpha_k(y) = \alpha_{i,j}^{(k)}(y)$ if y is in the polyhedron determined by $y_i y_j$ and $\alpha_k(y) = \alpha_{i_1, i_2, i_3}^{(k)}(y)$ if $y \in K_{i_1, i_2, i_3}$. Define $p_k \in C(Y, M_n)$ by $p_k(y_j) = p_k^j$, $p_k(y) = p_k^{i,j}(y)$ if $y \in K_{i,j}$ and $p_k(y) = p_k^{i_1, i_2, i_3}(y)$ if $y \in K_{i_1, i_2, i_3}$. Define $\psi : C(X) \rightarrow C(S(r), M_n)$ by

$$\psi(f) = \sum_{k=1}^n f(\alpha_k) p_k \text{ for all } f \in C(X). \quad (\text{e 6.135})$$

Note that $\psi(f)(t) = \Phi_{i_1, i_2, i_3}(f)(t)$ if $t \in K_{i_1, i_2, i_3}$ and $\psi(f)(t) = \sum_{k=1}^n f(\alpha_{i,j})(t) p_k^{i,j}(t)$ if $t \in K_{i,j}$. Moreover,

$$\psi(f)(y_j) = \varphi(f)(y_j) \text{ for all } f \in C(X), \quad j = 1, 2, \dots, K, \text{ and} \quad (\text{e 6.136})$$

$$\|\psi(f)(y) - \psi(f)(y_j)\| < \epsilon/4 \text{ for all } f \in C(X) \quad (\text{e 6.137})$$

and for some j so that y is in a simplex with y_j as one of the vertex.

Now one changes r . To simplify notation, one may assume that $\text{diam}(Y) \leq 1$. One obtains a sequence of open covers $\mathcal{U}_m = \{B_j^{(m)}\} = \{B(y_j^{(m)}, d(y_j^{(m)}, r_m)/12) : j = 1, 2, \dots, K(m)\}$ (with $d(y, r_m) = d'(y)r_m$) such that: (i) the order of the cover is at most 2, and,
(ii)

$$r_{m+1} < \min\{\epsilon_m/2, \min\{d'(y_j^{(m)})/2^{m+1} : 1 \leq j \leq K(m)\}\}, \quad (\text{e 6.138})$$

where ϵ_m is a Lebesgue number for the cover \mathcal{U}_m . It follows from (ii) that (iii) holds: if $B_{j_1}^{(m+1)} \cap B_{j_2}^{(m+1)} \cap \dots \cap B_{j_l}^{(m+1)} \neq \emptyset$, then there exists $k \leq K(k)$ such that $B_{j_s}^{(m+1)} \subset B_k^{(m)}$, $s = 1, 2, \dots, l$. For each $m = 1, 2, \dots$, let \mathcal{S}_m be the simplicial complex constructed from points $\{y_1, y_2, \dots, y_{K(m)}\}$ as above, and let $S(r_m)$ be the underline polyhedra of dimension at most 2 (see (e 6.123) and (e 6.124)). Denote by $\psi_m : C(X) \rightarrow C(S(r_m), M_n)$ the unital homomorphism constructed above using $r = r_m$, $m = 1, 2, \dots$.

To specify the map $\pi_m^{m+1} : S(r_{m+1}) \rightarrow S(r_m)$, for each $j (\leq K(m+1))$, let $y_j^{(m+1)}$ be one of the vertex. By virtue of (iii) above, the family

$$\mathcal{U}_{j,m} = \{B_{j'}^{(m)} : B_j^{(m)} \subset B_{j'}^{(m)}\}$$

is non-empty. Since $\cap B_{j'}^{(m)} \in \mathcal{U}_{j,m} \neq \emptyset$, the vertices of S_m which correspond to the members of $\mathcal{U}_{m,j}$ span a simplex $K^{(j,m)} \in \mathcal{S}_m$. Define

$$\pi_m^{m+1}(y_j^{(m+1)}) = b(K^{(j,m)}), \quad (\text{e 6.139})$$

where $b(K^{(j,m)})$ denotes the barycenter of $K^{(j,m)}$. As in the proof 1.13.2 of [3], this implies that, for every simplex $S \in \mathcal{S}(r_{m+1})$, the images of vertices of S under π_m^{m+1} are contained in a simplex $T \in \mathcal{S}(r_m)$.

This construction leads to an inverse limits $\lim_{\leftarrow m} (S_{m+1}, \pi_m^{m+1})$ which is homeomorphic to Y (see the proof of 1.13.2 of [3]). One identifies these two spaces. Denote by $\pi_m^\infty : Y \rightarrow S_m$ the continuous map induced by the inverse limit.

Denote by $J_m : C(S(r_m), M_n) \rightarrow C(S(r_{m+1}), M_n)$ the unital homomorphism defined by

$$J_m(f)(y) = f(\pi_m^{m+1}(y)) \text{ for all } f \in C(S(r_m), M_n), \quad (\text{e 6.140})$$

$m = 1, 2, \dots$. Denote by $J_{m,\infty} : C(Y, M_n) \rightarrow C(S(r_m), M_n)$ the unital homomorphism induced by the inductive limit $C(Y, M_n) = \lim_{m \rightarrow \infty} (C(S(r_m), M_n), J_m)$ which can also be defined by $J_{m,\infty}(f)(y) = f(\pi_m^\infty(y))$ for all $f \in C(Y, M_n)$.

Fix $y \in Y$ and m . There is a simplex $K_m \in \mathcal{S}(r_m)$ such that $\pi_m^\infty(y) \in K_m$ and therefore there exists a vertex $y_{j(m)}^{(m)}$ such that

$$\text{dist}(y, y_{j(m)}^{(m)}) < d'(y_{j(m)}^{(m)})/6 \cdot 2^m \quad (\text{e 6.141})$$

(see for example the proof of 1.13.2 of [3]). Let $\psi_m : C(X) \rightarrow C(S(r_m), M_n)$ be the unital homomorphism construct above (by replacing r by r_m). So

$$\psi_m(f) = \sum_{k=1}^n f(\alpha_k) p_k \text{ for all } f \in C(X),$$

where α_k and p_k ($k = 1, 2, \dots, n$) as constructed above (with r replaced by r_m).

One estimates, by (e 6.141), (e 6.122), (e 6.136) and (e 6.137), that

$$\begin{aligned} \|\varphi(f)(y) - J_{m,\infty} \circ \psi_m(f)(y)\| &\leq \|\varphi(f)(y) - \varphi(f)(y_{j(m)}^{(m)})\| + \|\varphi(f)(y_{j(m)}^{(m)}) - \psi_m(f)(y_{j(m)}^{(m)})\| \\ &\quad + \|\psi_m(f)(y_{j(m)}^{(m)}) - \psi_m(f)(\pi_m^\infty(y))\| \\ &< \epsilon/16 + 0 + \epsilon/4 < \epsilon \end{aligned} \quad (\text{e 6.142})$$

for all $f \in \mathcal{F}$. Note that

$$J_{m,\infty} \circ \psi_m(f) = \sum_{k=1}^n f(\alpha_k \circ \pi_m^\infty) p_k \quad (\text{e 6.143})$$

for all $f \in C(X)$. This completes the proof. \square

Definition 6.4. Let Y be a compact metric space and $C \subset C(Y, M_n)$ be a unital C^* -subalgebra. C is said to be *diagonalized* if there are mutually orthogonal rank one projections $\{p_1, p_2, \dots, p_n\} \subset C(Y, M_n)$ such that p_i commutes with every element in C , $i = 1, 2, \dots, n$.

Theorem 6.5. Let X be a compact metric space and let $n \geq 1$. Suppose that Y is a compact metric space with $\dim Y \leq 2$ and suppose that $\varphi : C(X) \rightarrow C(Y, M_n)$ is a unital homomorphism. Then, for any $\epsilon > 0$ and any compact subset $\mathcal{F} \subset C(X)$, there is a unital commutative C^* -subalgebra $B \subset C(Y, M_n)$ which can be diagonalized and

$$\text{dist}(\varphi(f), B) < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e6.144})$$

Proof. One may view that $X \subset I$, where I is the Hilbert cube which is viewed as a subset of l^2 . Note that Hilbert cube is convex and every open (or closed) ball in I is convex and therefore is absolute retract and locally absolute retract. Thus $X = \bigcap_{m=1}^{\infty} F_m$, where each F_m is a finite union of closed balls of I . In particular, each F_m is locally absolute retract. Let $\iota_m : F_{m+1} \rightarrow F_m$ ($m = 1, 2, \dots$) and $\iota_m^\infty : X \rightarrow F_m$ be the embeddings. Let $j_m : C(F_m) \rightarrow C(F_{m+1})$ defined by $j_m(f) = f \circ \iota_m$ and $j_{m,\infty} : C(F_m) \rightarrow C(X)$ by $j_{m,\infty}^\infty(f) = f \circ \iota_m^\infty$ for all $f \in C(X)$. Now let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. For each $f \in \mathcal{F}$, there is $m \geq 1$ and $g_f \in C(F_m)$ such that

$$\|j_m^\infty(g_f) - f\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e6.145})$$

Let $\mathcal{G} = \{g_f : f \in \mathcal{F}\}$. By considering $\varphi_m = \varphi \circ j_m^\infty$, one obtains, by applying 6.3, continuous maps $\beta_i : Y \rightarrow F_m$ and mutually orthogonal rank one projections $p_1, p_2, \dots, p_n \in C(Y, M_n)$ such that

$$\|\varphi_m(g_f) - \sum_{i=1}^n g_f(\beta_i) p_i\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \quad (\text{e6.146})$$

By combining (e6.146) with (e6.145), one has that

$$\|\varphi(f) - \sum_{i=1}^n g_f(\beta_i) p_i\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

Let B be the commutative C^* -subalgebra generated by $\sum_{i=1}^n g_f(\beta_i) p_i$ for $f \in \mathcal{F}$. \square

Corollary 6.6. Let Y be a compact metric space with $\dim Y \leq 2$, let $n \geq 1$ be an integer and let x be a normal element. Then, there are n sequences of functions $\{\lambda_k^{(m)}\}$ in $C(Y)$ ($k = 1, 2, \dots, n$) and there is a sequence of sets of n mutually orthogonal rank one projections $\{p_1^{(m)}, p_2^{(m)}, \dots, p_n^{(m)}\} \subset C(Y, M_n)$ such that

$$\lim_{m \rightarrow \infty} \|x - \sum_{k=1}^n \lambda_k^{(m)} p_k^{(m)}\| = 0.$$

Moreover, if x is self-adjoint, $\lambda_k^{(m)}$ can be chosen to be real and if x is a unitary, $\lambda_k^{(m)}$ can be chosen so that $|\lambda_k^{(m)}| = 1$, $k = 1, 2, \dots, n$ and $m = 1, 2, \dots$

7 Higher dimensional cases

In this section, we consider the cases that $\dim Y \geq 3$. One would hope that the similar argument used in section 6 can repeat for higher dimensional space Y . In fact, a version of 5.1 and 5.2 can be proved for two dimensional spaces. However, the last requests in 5.1 and 5.2 can not be improved, for example, in a generalized 5.2, $U(s)$ can not be chosen so it exactly commutes with φ on a given line segment even u can. The reason is that not every homomorphism to $C([0, 1], M_n)$ can be exactly diagonalized (see [9]). This technical problem is fatal as one can see from the results of this section. Nevertheless, one has the following.

Proposition 7.1. *Let X be a zero dimensional compact metric space, $n \geq 1$ and let Y be a compact metric space for which every minimal projection in $C(Y, M_n)$ has rank one or zero at each point of Y (which is the case if $\dim Y \leq 3$ —see 7.2).*

Then any unital homomorphism $\varphi : C(X) \rightarrow C(Y, M_n)$ can be approximately diagonalized.

Proof. We may assume that $X \subset \mathbb{R}$. Choose an element $x \in C(X)$ with $sp(x) = X$. If $f \in C(X)$ then one has $f = f(x)$. For any $\delta > 0$, there are mutually orthogonal projections $\{e_1, e_2, \dots, e_N\} \subset C(X)$ for which $\sum_{j=1}^N e_j = 1$ and real numbers $\lambda_1, \lambda_2, \dots, \lambda_N \in X$ such that

$$\|x - \sum_{j=1}^N \lambda_j e_j\| < \delta. \quad (\text{e 7.147})$$

Given a finite subset $\mathcal{F} \subset C(X)$ and $\epsilon > 0$, by choosing a sufficiently small δ , one may assume that

$$\|f(x) - \sum_{j=1}^N f(\lambda_j) e_j\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.148})$$

Let $\varphi : C(X) \rightarrow C(Y, M_n)$ be a unital homomorphism. Then

$$\|\varphi(f) - \sum_{j=1}^N f(\lambda_j) \varphi(e_j)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.149})$$

Since each $\varphi(e_j)$ is a projection, by the assumption, there are mutually orthogonal projections $\{p_{j,1}, p_{j,2}, \dots, p_{j,r(j)}\} \subset C(Y, M_n)$ such that $p_{j,i}(y)$ has rank either one or zero and $\sum_{i=1}^{r(j)} p_{j,i} = \varphi(e_j)$, $j = 1, 2, \dots, N$. Since $\sum_{j=1}^N \varphi(e_j) = 1_{C(Y, M_n)}$, it is easy to find mutually orthogonal rank one projections p_1, p_2, \dots, p_n , where each p_i is a sum of some projections $p_{j,k}$, such that $\sum_{i=1}^n p_i = 1_{C(Y, M_n)}$. Suppose that $p_i = \sum_k p_{j_k, i_k}$, $n = 1, 2, \dots, n$. Let Y_{j_k, i_k} be the clopen set so that $p_{j_k, i_k}(y) \neq 0$. Define a continuous map $\alpha_i : Y \rightarrow X$ by $\alpha_i(y) = \lambda_{j_k}$ if $y \in Y_{j_k, i_k}$ and $\alpha_i(y) = 0$ if y is not in the support of p_i . Then, by (e 7.149),

$$\|\varphi(f) - \sum_{i=1}^n f(\alpha_i) p_i\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 7.150})$$

□

Remark 7.2. Let Y be a connected finite CW complex with $\dim Y = d \leq 3$. Then $[d/2] \leq 1$. Let $p \in M_n(C(Y))$ be a projection with rank $r \geq 2$. It follows from 8.12 of [6] that there exists a projection $q \leq p$ with rank $r - [d/2]$ which is trivial. Thus $q \neq 0$. Then there is a rank one projection $e \leq q \leq p$. Therefore p is not minimal. This shows that any minimal projection

of $M_n(C(Y))$ has rank one. Since every compact metric space is an inverse limit of finite CW complex, it is easy to see that, if Y is a compact metric space with $\dim Y \leq 3$ and $p \in M_n(C(Y))$ is a minimal projection, then $p(y) = 0$, or $p(y)$ has rank one for any $y \in Y$.

Let Y be a compact metric space with $\dim Y > 3$. Suppose that $C(Y, M_n)$ contains a minimal projection p with rank at least 2. Suppose that X is not connected, say X is a disjoint union of two clopen subsets X_1 and X_2 . Choose a unital homomorphism $\varphi_1 : C(X_1) \rightarrow (1-p)C(Y, M_n)(1-p)$ and a unital homomorphism $\varphi_2 : C(X_2) \rightarrow pC(Y, M_n)p$. Define $\varphi : C(X) \rightarrow C(Y, M_n)$ by $\varphi(f) = \varphi_1(f|_{X_1}) + \varphi_2(f|_{X_2})$ for all $f \in C(X)$. Then φ can not be possibly approximately diagonalized because p is a minimal projection. Therefore, in what follows, we mainly consider the case that X is connected, or, at least the case that $\dim X \geq 1$.

Proposition 7.3. *Let Y be a compact metric space for which $\pi^1(Y)$ is trivial and $K_1(C(Y)) \neq \{0\}$. Then there are unital homomorphisms from $C(\mathbb{T}) \rightarrow C(Y, M_n)$ for some $n \geq 2$ which can not be approximately diagonalized.*

Proof. Since $K_1(C(Y)) \neq \{0\}$, there is an integer $n \geq 2$ and a unitary $u \in C(Y, M_n)$ such that $u \notin U_0(C(Y, M_n))$. Define a unital homomorphism $\varphi : C(\mathbb{T}) \rightarrow C(Y, M_n)$ by $\varphi(f) = f(u)$ for all $f \in C(\mathbb{T})$. Suppose that there are continuous maps $\alpha_k : Y \rightarrow \mathbb{T}$, $k = 1, 2, \dots, n$ and mutually orthogonal rank one projections $\{p_1, p_2, \dots, p_n\} \subset C(Y, M_n)$ such that

$$\|\varphi(z) - \sum_{k=1}^n z(\alpha_k)p_k\| < 1, \quad (\text{e 7.151})$$

where z is the identity function on the unit circle \mathbb{T} . Note that $u = \varphi(z)$. Since $\pi^1(Y) = \{0\}$, for each k , there is a continuous path of unitaries $\{w_k(t) : t \in [0, 1]\} \subset C(Y)$ such that

$$w_k(0) = z(\alpha_k) \text{ and } w_k(1) = 1.$$

One defines a continuous path of unitaries $\{U(t) : t \in [0, 1]\} \subset U(C(Y, M_n))$ by

$$U(t) = \sum_{k=1}^n w_k(t)p_k \text{ for } t \in [0, 1].$$

Then $U(0) = \sum_{k=1}^n z(\alpha_k)p_k$ and $U(1) = 1_{M_n}$. So $\sum_{i=1}^n z(\alpha_k)p_k \in U_0(C(Y, M_n))$. By (e 7.151), $u \in U_0(C(Y, M_n))$. A contradiction. \square

Corollary 7.4. *There is a unital homomorphism $\varphi : C(\mathbb{T}) \rightarrow C(S^3, M_2)$ such that φ can not be approximately diagonalized.*

Proof. Let

$$u(z, w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

where $(z, w) \in S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. Then $u \in U(M_2(C(S^3))) \setminus U_0(M_2(C(S^3)))$. However, $\pi^1(S^3) = \{0\}$. Thus, as in the proof of 7.3, this unitary can not be approximated by unitaries which are diagonalized. \square

Corollary 7.5. *Let X be a finite CW complex with $\dim X \geq 2$ and let Y be a compact metric space for which $\pi^1(Y)$ is trivial but $K_1(C(Y))$ is not trivial. Then there are unital homomorphism $\varphi : C(X) \rightarrow C(Y, M_n)$ for some $n \geq 2$ which can not be approximately diagonalized.*

Proof. X contains a subset D which is homeomorphic to the unit disk. Thus X contains a subset S which is homeomorphic to \mathbb{T} . Define $s : C(X) \rightarrow C(\mathbb{T})$ by the restriction on S and then take the homeomorphism. If $\varphi : C(\mathbb{T}) \rightarrow C(Y, M_n)$ is one of those unital homomorphisms which can not be diagonalized (by 7.3), then $\varphi \circ s$ can not be diagonalized. \square

Lemma 7.6. *For any $d > 0$, if $0 < \delta < d$, and if a and b are two normal elements in a unital C^* -algebra such that*

$$\|a - b\| < \delta,$$

then

$$sp(b) \subset \{\lambda \in \mathbb{C} : \text{dist}(\lambda, sp(a)) < d\}.$$

The proof is standard and known. The point here is that δ does not depend on a and b .

Lemma 7.7. *Let Y be a compact metric space and let $g \in C(Y, M_2)$ be a normal element for which $sp(g(y)) = \{\lambda(y), \overline{\lambda(y)}\}$ for each $y \in Y$. For any $\epsilon > 0$, if $\alpha_1, \alpha_2 : Y \rightarrow \mathbb{C}$ are two continuous maps and if $p_1, p_2 \in C(Y, M_2)$ are mutually orthogonal rank one projections such that*

$$\|g - (\alpha_1 p_1 + \alpha_2 p_2)\| < \epsilon/8, \tag{e 7.152}$$

then

$$\|g - (\alpha_1 p_1 + \overline{\alpha_1} p_2)\| < \epsilon. \tag{e 7.153}$$

Proof. Suppose that (e 7.152) holds. It follows from 7.6 that, for each $y \in Y$,

$$|\lambda(y) - \alpha_1(y)| < \epsilon/7 \text{ or } |\overline{\lambda(y)} - \alpha_1(y)| < \epsilon/7 \tag{e 7.154}$$

Then

$$|\overline{\lambda(y)} - \alpha_2(y)| < \epsilon/7 \text{ or } |\lambda(y) - \alpha_2(y)| < \epsilon/7. \tag{e 7.155}$$

It follows that

$$|\alpha_2(y) - \overline{\alpha_1(y)}| < 2\epsilon/7 \tag{e 7.156}$$

for all $y \in Y$. Therefore

$$\|g - (\alpha_1 p_1 + \overline{\alpha_1} p_2)\| < \epsilon. \tag{e 7.157}$$

\square

By modifying 4.4 of [9], one has the following:

Lemma 7.8. *Let Y be a finite CW complex with $\dim Y > 3$. Then there is a self-adjoint element $b \in C(Y, M_2)$ with $sp(b) = [-1, 1]$ which can not be approximately diagonalized.*

Proof. Since Y is a finite CW complex with $\dim Y > 3$, it contains a 4-dimensional cube. Therefore there is a subset $Y_0 \subset Y$ such that Y_0 is homeomorphic to S^3 . Identifying Y_0 with S^3 , one obtains a unitary $u \in U(M_2(C(Y_0))) \setminus U_0(M_2(C(Y_0)))$ which has the form

$$u(z, w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

where $(z, w) \in S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.

In fact, for every $y \in Y_0$, $u(y) \in SU(2)$. Since $SU(2)$ is absolute neighborhood retract, there is a neighborhood Y_1 of Y_0 and a map $U \in C(Y_1, SU(2))$ which extends u . Let $f \in C(Y)$ be a function such that $0 \leq f(y) \leq 1$, $f(y) = 1$ if $y \in Y_0$ and $f(y) = 0$ if $y \in Y \setminus Y_1$. Define a normal element $g \in C(Y, M_2)$ by $g(y) = f(y)U(y)$ if $y \in Y_1$ and $g(y) = 0$ if $y \in Y \setminus Y_1$. Note that the eigenvalues of g have the form λ and $\bar{\lambda}$, where $|\lambda|^2 = f(x)^2$. Let $1/3 > \delta > 0$. Suppose that there are continuous maps $\alpha_1, \alpha_2 : Y \rightarrow D$, where D is the unit disk, and mutually orthogonal rank one projections $p_1, p_2 \in C(Y, M_2)$ such that

$$\|g - \sum_{i=1}^2 \alpha_i p_i\| < \delta/8. \quad (\text{e 7.158})$$

It follows from 7.7 that

$$\|g - (\alpha_1 p_1 + \overline{\alpha_1} p_2)\| < \delta. \quad (\text{e 7.159})$$

Let $\pi : C(Y, M_2) \rightarrow C(Y_0, M_n)$ be the quotient map. Then

$$\|u(y) - (\alpha_1(y)\pi(p_1) + \overline{\alpha_1(y)}\pi(p_2))\| < \delta \text{ for all } y \in Y_0. \quad (\text{e 7.160})$$

Since, for each $y \in Y_0$, $u(y)$ has eigenvalues $\lambda(y)$ and $\overline{\lambda(y)}$ with $|\lambda(y)| = 1$, with small δ , one may assume that

$$|\alpha_1(y) - \alpha_1(y)|\alpha_1(y)|^{-1}| < 1/8 \text{ for all } y \in Y_0,$$

$i = 1, 2$. It follows (with a sufficiently small δ) that

$$\|u(y) - (\beta_1(y)\pi(p_1) + \overline{\beta_1(y)}\pi(p_2))\| < 1/2 \quad (\text{e 7.161})$$

for all $y \in Y_0$, where $\beta_1(y) = \alpha_1(y)|\alpha_1(y)|^{-1}$. Since β_1 maps Y_0 to S^1 , Y_0 is homeomorphic to S^3 and since $\pi^1(S^3) = \{0\}$, there is a continuous path of $\{w(t) : t \in [0, 1]\} \subset U(C(Y_0, M_2))$ such that

$$w(0) = \beta_1 \text{ and } w(1) = 1. \quad (\text{e 7.162})$$

Thus $\beta_1(y)\pi(p_1) + \overline{\beta_1(y)}\pi(p_2) \in U_0(C(Y_0, M_2))$. From (e 7.161) and the fact that $u \notin U_0(C(Y_0, M_2))$, this is impossible.

Therefore g can not be approximately diagonalized. On the other hand, $g(y) + g^*(y) = \gamma(y)1_{M_2}$, where $\gamma(y) = \lambda(y) + \bar{\lambda}(y)$, for all $y \in Y$. Let $b = (1/2i)(g - g^*)$. Then b is self-adjoint. Suppose that there were sequences $\alpha_{j,n} : Y \rightarrow \mathbb{R}$ and sequences of pairs $\{p_{1,n}, p_{2,n}\}$ of mutually orthogonal rank one projections in $C(Y, M_2)$ such that

$$\lim_{n \rightarrow \infty} \|b - (\sum_{j=1}^2 \alpha_{j,n} p_{j,n})\| = 0. \quad (\text{e 7.163})$$

Note that $p_{1,n} + p_{2,n} = 1_{M_2}$. Let $\beta_{j,n} = \gamma + \sqrt{-1}\alpha_{j,n}$, $j = 1, 2$ and $n = 1, 2, \dots$. It would imply that

$$\lim_{n \rightarrow \infty} \|g - \sum_{j=1}^2 \beta_{j,n} p_{j,n}\| = 0. \quad (\text{e 7.164})$$

This contradicts what we have proved that g can not be approximately diagonalized. Therefore one concludes that the self-adjoint element b can not be approximately diagonalized.

Finally, since $u \in U(C(S^3, M_2)) \setminus U_0(C(S^3, M_2))$, $sp(u) = \mathbb{T}$. It follows that $sp(b) = [-1, 1]$. \square

Theorem 7.9. *Let Y be a finite CW complex with $\dim Y > 3$ and let $n \geq 2$ be an integer. Then, for any finite CW complex X with $\dim X \geq 1$, there exists a unital homomorphism $\varphi : C(X) \rightarrow C(Y, M_n)$ which can not be approximately diagonalized.*

Proof. X contains a compact subset which is homeomorphic to $[-1, 1]$. There is a surjective homomorphism $s : C(X) \rightarrow C([-1, 1])$. If $n = 2$, let $b \in C(Y, M_2)$ with $sp(b) = [-1, 1]$ be the self-adjoint element given by 7.8 which can not be approximated by self-adjoint elements in $C(Y, M_2)$ which can be diagonalized. Define $\varphi : C(X) \rightarrow C(Y, M_2)$ by $\varphi(f) = s(f)(b)$ for all $f \in C(X)$. Then φ can not be approximated by homomorphisms which can be diagonalized.

If $n \geq 3$, let $\gamma : [-1, 1] \rightarrow [0, 1/2] \cup \{1\}$ be a homeomorphism. Put $\Omega_1 = [0, 1/2] \cup \{1\}$. Let $\{e_{i,j} : 1 \leq i, j \leq n\}$ be a system of matrix units. Define $\varphi_1 : C(\Omega_1) \rightarrow C(Y, M_n)$ by

$$\varphi_1(f) = f(b)(e_{11} + e_{22}) + f(1)\left(\sum_{i=3}^n e_{ii}\right) \text{ for all } f \in C(\Omega_1). \quad (\text{e 7.165})$$

Suppose that φ_1 that there exist continuous maps $\alpha_{j,k} : X \rightarrow \Omega_1$ ($j = 1, 2, \dots, n$) and mutually orthogonal rank one projections $p_{1,k}, p_{2,k}, \dots, p_{n,k}$, $k = 1, 2, \dots$, such that

$$\lim_{k \rightarrow \infty} \left\| \varphi_1(f) - \sum_{j=1}^n f(\alpha_{j,k}) p_{j,k} \right\| = 0 \quad (\text{e 7.166})$$

for all $f \in C(\Omega_1)$. Denote $\psi_k(f) = \sum_{j=1}^n f(\alpha_{j,k}) p_{j,k}$ for all $f \in C(\Omega_1)$.

Let $f_0 \in C(\Omega_1)$ such that $f_0(t) = 0$ if $t = 1$ and $f_0(t) = 1$ if $t \in [0, 1/2]$. Then f_0 is a projection. It follows that

$$\lim_{k \rightarrow \infty} \left\| \varphi_1(f_0) - \psi_k(f_0) \right\| = 0. \quad (\text{e 7.167})$$

Note that $\varphi_1(f_0) = e_{11} + e_{22}$. From (e 7.167), when k is sufficiently large, there are unitaries $v_k \in C(Y, M_n)$ such that

$$v_k^* \psi_k(f_0) v_k = e_{11} + e_{22} \text{ and } \lim_{k \rightarrow \infty} \|1 - v_k\| = 0. \quad (\text{e 7.168})$$

Since $\psi_k(f_0)$ is a projection which commutes with $p_{j,k}$ and $p_{j,k}$ is a rank one projection, it follows that $\psi_k(f_0)p_{j,k} = p_{j,k}$ or $\psi_k(f_0)p_{j,k} = 0$. By (e 7.168), $\psi_k(f_0)$ has rank 2 for all sufficient large k . To simplify notation, one may assume that

$$\psi_k(f_0)p_{j,k} = p_{j,k}, \quad j = 1, 2.$$

Let $q_{j,k} = v_k^* \psi_k(f_0) v_k$, $j = 1, 2$, $k = 1, 2, \dots$. Note that $q_{j,k}$ has rank one, $j = 1, 2$. By (e 7.166), (e 7.167) and (e 7.168),

$$\lim_{k \rightarrow \infty} \left\| \varphi_1(f f_0) - \sum_{j=1}^2 f(\alpha_{j,k}) q_{j,k} \right\| = 0 \quad (\text{e 7.169})$$

for all $f \in C(\Omega_1)$. This would imply that b could be approximated by diagonalizable self-adjoint elements. A contradiction.

Therefore φ_1 can not be approximated by homomorphisms which are diagonalizable. There is a surjective homomorphism $s_1 : C(X) \rightarrow C(\Omega_1)$. Define $\varphi : C(X) \rightarrow C(Y, M_n)$ by $\varphi(f) = \varphi_1(s_1(f))$ for all $f \in C(X)$. Then φ can not be approximated by homomorphisms which can be diagonalized. □

References

- [1] O. Bratteli, G. A. Elliott, D. Evans and A. Kishimoto, *Homotopy of a pair of approximately commuting unitaries in a simple C^* -algebra*, J. Funct. Anal. **160** (1998), 466–523.
- [2] M-D. Choi and E. Effros, *The completely positive lifting problem for C^* -algebras*, Ann. of Math. **104** (1976), 585–609.
- [3] R. Engelking, *Dimension Theory*, North-Holland Publishing (1978)
- [4] G. Gong and H. Lin, *Almost multiplicative morphisms and almost commuting matrices*, J. Operator Theory **40** (1998), 217–275.
- [5] P. R. Halmos and H. E. Vaughan, *The marriage problem*, Amer. J. Math. **72**, (1950). 214–215.
- [6] D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966, reprinted in Springer-Verlag Graduate Texts in Mathematics.
- [7] R. V. Kadison, *Diagonalizing matrices over operator algebras*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 84–86.
- [8] R. V. Kadison, *Diagonalizing matrices*, Amer. J. Math. **106** (1984), 1451–1468.
- [9] K. Grove and G. K. Pedersen, *Diagonalizing matrices over $C(X)$* , J. Funct. Anal., **59** (1984), 65–89.
- [10] H. Lin, *Almost commuting selfadjoint matrices and applications. Operator algebras and their applications* (Waterloo, ON, 1994/1995), 193–233, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.
- [11] H. Lin, *An introduction to the classification of amenable C^* -algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xii+320 pp. ISBN: 981-02-4680-3
- [12] H. Lin, *Approximate homotopy of homomorphisms from $C(X)$ into a simple C^* -algebra*, Mem. Amer. Math. Soc., to appear (arxiv.org/ OA/0612125).
- [13] H. Lin, *Homotopy of unitaries in simple C^* -algebras with tracial rank one*, preprint, arXiv:0805.0583
- [14] W.A.J. Luxembourgh, F.R. Taylor, *Almost commuting matrices are near commuting matrices*, Indag. Math. **32**(1970), 96–98.
- [15] C. Pearcy, A. Shields, *Almost commuting matrices*, J. Funct. Anal. **33** (1979), 332–338.
- [16] G. K. Pedersen, *SAW*-algebras and corona C^* -algebras, contributions to noncommutative topology*, J. Operator Theory **15** (1986), 15–32.

email: hlin@uoregon.edu